GAUGE AND GRAVITY SCATTERING AMPLITUDES FROM CHY FORMALISM

by

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ABSTRACT

In this dissertation, we first review some recent progress on exploring the nature of scattering amplitudes. Then we present our recent work on direct evaluation of tree level maximally helicity violating (MHV) amplitudes by Cachazo-He-Yuan (CHY) formula, which naturally reproduce the Parke-Taylor and Hodges formula, respectively, for gauge and gravity. We also verify that they are supported only by one single solution to the scattering equation. In addition, we derive a new compact formula for tree level single trace MHV amplitudes for Einstein-Yang-Mills theory, which is equivalent to, but much simpler than the known Selivanov-Bern-De Freitas-Wong (SBDW) formula. It can be shown that other solutions do not contribute to the MHV amplitudes of Yang-Mills, gravity and Einstein-Yang-Mills theory. We further propose a method to characterize the solutions to the scattering equations using the rank of two discriminant matrices. In four dimensions, such a characterization can be used to understand the correspondence between the helicity configurations of external scattering particles and the solutions to the scattering equation.

For my family.

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NOTATION AND SYMBOLS

$\eta_{\mu\nu}$	flat space-time metric with the most-negative signature
$g_{\mu\nu}$	general space-time metric with the most-negative signature
p^μ, k^μ, q^μ, etc	space-time momentum
ϵ^μ	polarization vector
$(\sigma^\mu)_{\alpha\dot{\alpha}}$	four-vector composed of the 2×2 Pauli matrices $(1, \sigma^i)$
$(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}$	four-vector composed of the 2×2 Pauli matrices $(1, -\sigma^i)$
$\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$	2×2 antisymmetric matrices: $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon_{12} = -\epsilon_{21} = -1$
λ_α	left-handed Weyl two-spinor
$\widetilde{\lambda}_{\dot{\alpha}}$	right-handed Weyl two-spinor
s_{ab}	Mandelstam variable, $s_{ab} = (p_a + p_b)^2$
$M^{i_1 \ldots i_r}_{j_1 \ldots j_r}$	a submatrix of M with rows $\{i_1 \ldots i_r\}$ and columns $\{j_1 \ldots j_r\}$ deleted
M_{IJ}	a submatrix of M with row and column index ranging within I and J

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CHAPTER 1

INTRODUCTION

Relativistic quantum field theory is undoubtedly a big triumph in the last century. It not only sets the playground for the high-energy particle physics, but also finds numerous applications in many other branches of modern physics. The nonabelian gauge theory (or Yang-Mills theory) is the cherry on the cake, due to its mathematical beauty and pivotal role in describing how fundamental particles interact in the Standard Model.

On the experimental side, studying how particles scatter with each other at colliders plays a very crucial role in test existing theories and probing new physics. One of the most important physical observables in a scattering experiment is the differential cross section, which describes the angular dependence of the scattering probability. This quantity provides a link between theory and experiment. Thus, calculating the cross section accurately is of paramount importance since the correct interpretation of experimental results at colliders hinges on it.

In the perturbative regime of field theory, the cross section can be obtained through a phase space integral over the norm-square of the scattering amplitude. Because of its importance, the scattering amplitude has been a main subject of research ever since the birth of quantum field theory. In the 1970s, this effort seemed to conclude with the Lagrangian formalism of scattering amplitudes. The scheme is as follows:

1. Fix the gauge freedom in the Lagrangian using the Fadeev-Popov trick;
2. Read off the Feynman rules of each interaction vertex and propagator;
3. At the leading order, sum over all tree level Feynman diagrams;
4. At higer orders, sum over all loop diagrams with the loop momentum integration properly regularized;
5. Renormalize the theory by absorbing the divergence into the counterterms.

Nowadays, most quantum field theory textbooks, for example [1], follow this scheme.

However, people soon realized that the computational complexity grew factorially with the number of external particles, especially for gauge interactions. This is not so alarming since many-particles scattering are expected to be complicated by then.

In the mid-1980s, people started to discuss the concept of the hadron collider. Subsequently, understanding the gluon[1] scattering became very important since it dominated the hadron jet production at the collider. Parke and Taylor first calculated the gluon two-to-four scattering amplitude at the tree level. In their eleven-page paper [2], the result spanned about eight pages. With such a messy outcome, they concluded that this result is "suitable for fast numerical calculations." However, at the end of the paper, they still hoped to "obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight." The effort indeed paid off. They soon realized that the norm-square of the amplitude, which is a gauge invariant quantity, was much simpler. In a famous paper [3], they proposed the following formulas for arbitrary n-point color ordered gluon tree level amplitude (to be discussed in Section 2.2):

$$\begin{aligned} \left|A_n(1^+2^+3^+\dots n^+)\right|^2 &= \left|A_n(1^-2^+3^+\dots n^+)\right|^2 = 0 \\ \left|A_n(1^-2^-3^+\dots n^+)\right|^2 &= \frac{(p_1\cdot p_2)^4}{(p_1\cdot p_2)(p_2\cdot p_3)\dots(p_n\cdot p_1)}\,, \end{aligned} \tag{1.1}$$

where all particles are assumed to be outgoing. When we gradually flip the helicity configuration away from the uniform, $A_n(1^-2^-3^+\dots n^+)$ is the first nonzero amplitude, which is called the maximally helicity violating (MHV) amplitude. Parke and Taylor's confidence in this simple result comes from the correct collinear limit behavior. Meanwhile, Xu, Zhang, and Chang developed the spinor helicity formalism for massless momentum and polarization vector [4]. Using this later called "Chinese magic" variable (to be discussed in Section 2.1), Mangano, Parke, and Xu put the gluon MHV amplitude itself into an explicit gauge invariant form [5]:

$$A_n(1^-2^-3^+\dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\,. \tag{1.2}$$

This form, which is effectively the "square-root" of Eq. (1.1), is nowadays referred as the Parke-Taylor formula. This discovery opens the field of modern scattering amplitude.

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[1] The gluon here stands for general $SU(N)$ gauge boson, not just the phenomenologically important $SU(3)$ gauge boson that mediates the strong interaction.

The surprising simplicity in the Parke-Taylor formula strongly suggests that the complexity in the traditional Feynman diagram approach largely comes from the fact that it fails to capture the most essential structures of scattering amplitudes. In hindsight, the reason is twofold. First, the Feynman diagram approach insists on off-shell gauge invariance and locality, while physical amplitudes are all on-shell. As a result, the off-shell calculation involves a lot of unphysical baggage only to be cancelled when all the external particles are taken on-shell. If we can develop an algorithm that involves only on-shell quantities, the overly complicated Feynman rules and the massive summation over Feynman diagrams can be avoided. The second reason is that tree level gluon amplitudes can be embedded into $\mathcal{N} = 4$ superamplitudes. Then it turns out that there are some hidden symmetries (dual conformal and Yangian) that constrain the possible forms of amplitudes. These symmetries are not reflected in the Lagrangian.

The unnecessarily complicated calculation following the Feynman diagrams does not only plague the gauge theory, but it also plagues more severely in gravity. Following the traditional quantization prescription, the graviton three-point and four-point vertex involve 171 and 2850 terms, respectively [6]. Moreover, as a nonrenormalizable theory, there are an infinite number of higher-point contact terms. Consequently, extracting a meaningful result for graviton amplitudes using Feynman diagrams is extremely difficult. String theory provides a solution: to construct a closed string, one can just glue two open string together. Following this intuition, Kawai, Lewellen, and Tye found that a closed string amplitude can also be obtained by two open string amplitudes [7]. Subsequently, in the field theory limit, this relation stated that a gravity amplitude can be obtained by two color ordered gluon amplitudes with different particle orders. This discovery, later called the KLT relation (to be discussed in Section 2.3.2), indicated that although the perturbative gravity and gauge theory shared no similarity in the Lagrangian, they were secretly related through a “double copy.” The Lagrangian again failed to make such a connection.

The above discussion hints that the traditional Lagrangian formalism might not be suitable to describe the structure of gauge and gravity scattering amplitudes. In Section 1.1, we give a brief introduction to the modern ways of calculating amplitudes, with no reference to the Lagrangian. Then in Section 1.2, we give an outline on our work in this field, which will be discussed in detail in the following chapters.

1.1 Modern development

The discovery of Parke-Taylor formula and KLT relation motivated people to break the shackle of the Lagrangian formalism of scattering amplitudes. A more promising approach would be to study the mathematical structure of the amplitudes directly to gain a deeper understanding on the nature of field theory. In the following section, we provide a quick survey on important developments of modern scattering amplitude in the past three decades. We emphasize new and relevant development that led to the Cachazo-He-Yuan formalism for tree level amplitudes. We refer the readers to the textbooks [8, 9] for a comprehensive treatment of other interesting subjects.

After the Parke-Taylor formual was proposed, Berends and Giele (BG) abandoned the Feynman diagrams and developed a recursive algorithm [10] to construct the n-point tree amplitude with only one particle off-shell. Although from today's point of view, BG recursive relation does not embrace the full on-shell simplicity, it does provide an efficient way to attack higher-point amplitudes and is still of use today. With the help of BG recursive relation, Kleiss and Kuijf found the so-called KK relation among the color ordered gluon tree amplitudes [11]. This discovery reduces the number of independent color ordered amplitudes to from $(n-1)!$ to $(n-2)!$. In the 1990s, the unitarity cut method for loop integrand was developed [12, 13]. The basic idea is that when some internal loop momenta are on-shell, unitarity guarantees that the loop integrand reduces to the product of tree amplitudes. Then, by using the knowledge of tree amplitudes, we can reproduce certain coefficients of the loop integrand. In this way, we can avoid calculating the integrand from Feynman rules, which involves even more complicated calculation than the trees.

The previous mentioned development was motivated by the pursuit of fast numerical calculation of hadron jet production at the Large Hadron Collider (LHC). Along a different line of thought, string theorists also found interesting structures contained in the Parke-Taylor formula. Nair embedded the gluons into the $\mathcal{N}=4$ super Yang-Mills theory and noticed that the Parke-Taylor formula could be reproduced by a string correlation function [14].

Interestingly, the target space of this string theory is the space of supertwistor.[2] Although Nair's approach cannot reproduce the amplitudes beyond MHV, the insight on supertwistor string inspired a series of important breakthroughs fifteen years later. In 2003, Nair's expectation was realized by Witten, who found that general $\mathcal{N} = 4$ super Yang-Mill tree amplitude can be calculated from the correlation function of a topological string theory in the supertwistor space [17]. This work, together with a later paper by Roiban, Spradlin, and Volovich [18], proposed the Witten-RSV formalism for $\mathcal{N} = 4$ super Yang-Mills amplitude in four dimensions (all the terminologies will be defined later in this dissertation):

1. General N^kMHV sector superamplitude is supported by a set of degree $d = k+1$ polynomial curves in the supertwistor space;
2. Those polynomial curves are determined by solving a set of polynomial equations with external momentum data as input;
3. The entire permutation information is encoded in a Parke-Taylor-like factor:

$$PT(12\ldots n) = \frac{1}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)\ldots(\sigma_n - \sigma_1)}\,.$$

The set $\{\sigma\}$ is a solution to the above-mentioned polynomial equation. Quite remarkably, the KK relation is realized just as a complex number identity of PT.

This work shifts our understanding on the nature of scattering amplitude. It soon motivates the CSW [19] and BCFW [20] on-shell recursive relation for amplitudes. On one hand, these new recursive relations only use on-shell gauge invariant quantities, with no extra unphysical baggage. Therefore, the computational complexity is significantly reduced. The BCFW recursion is indeed intensively used at LHC to calculate jet production. On the other hand, the on-shell recursive relations put the amplitudes into new forms. Using the dual momentum variables, the BCFW construction naturally put the $\mathcal{N} = 4$ superamplitudes into an explicit dual superconformal invariant form [21,22]. However, from the Lagrangian, one can only tell the superconformal symmetry. Indeed, failing to realize a symmetry in the problem usually causes unnecessary complications. Moreover, superconformal and dual superconformal algebra generate an infinite dimensional Yan-

[2]The twistor is first introduced by Penrose [15], while the supersymmetrization is first written down by Ferber [16].

gian algebra [23], which has some implication on the integrability of the $\mathcal{N} = 4$ super Yang-Mills. Another feature of the amplitudes obtained from BCFW is that spurious poles exist and cancel each other, such that the amplitude is still a local quantity. Hodges first realized a geometric picture behind this cancellation [24]: the amplitude corresponds to the volume of a polytope, while BCFW produces a summation over the constituent simplices of this polytope. Then those spurious poles are just the internal boundaries in the triangulation of the polytope. This geometric interpretation was further generalized into the amplituhedron [25]. The BCFW can be also used to construct the loop integrand for $\mathcal{N} = 4$ super Yang-Mills. This effort finally culminated in the Grassmannian and on-shell diagram formalism, which is claimed to be valid at all loops [26]. We note that the Witten-RSV style twistor based formula for supergravity have been established in [27, 28].

Alternatively, Bern, Carrasco and Johansson [29, 30] proposed an very impressive construction for gravity amplitudes, which is now referred as the BCJ double copy relation. This pursuit starts with an implausible requirement at the first glance: to find a set of kinematic numerators of gauge amplitudes that satisfy the same Jacobi identity as the color factor (color kinematic duality). Surprisingly, such numerators can always be constructed. Once it is completed, by simply replacing the color factor by another copy of the kinematic numerator, we can obtain gravity from gauge amplitudes! The reason why those color kinematic duality satisfying numerators exist is interesting. The gauge amplitudes actually satisfy a new set of amplitude relations: the BCJ relation, which reduces the number of independent color ordered amplitudes from $(n-2)!$ (after using KK relation) to $(n-3)!$. This equivalence will be explored in Section 2.4.

As we previously mentioned, in the Witten-RSV formula, the only quantity that knows the amplitude relations is the Parke-Taylor factor PT. It turns out that on the polynomial curves that support amplitudes, the BCJ relation is indeed satisfied. On the other hand, we can impose PT to satisfied the BCJ relation. Then surprisingly, this condition alone can reproduce all the polynomial curves. Cachazo first indicated this point in [31], and later with He and Yuan, proposed the scattering equation [32, 33], which is equivalent to the condition that PT should satisfy BCJ. The benefit is that the scattering equation takes Mandelstam variables as input such that it should hold in arbitrary dimensions. Indeed, both KK and BCJ relation can be inferred from Feynman rules (although cumbersome), they should be generally true in any dimensions. Starting form 2013, Cachazo, He and Yuan (CHY) published a series of papers on the integrands for various field theory amplitudes [34–37]. The importance of the CHY formalism lies in that it extends the modern on-shell method to a spectrum of field theories beyond the gauge and gravity, for example, nonlinear sigma model, Born-Infeld, etc. Second, CHY is independent of the spinor helicity formalism, which provides the first on-shell formalism for amplitudes in arbitrary dimensions.

The following is an outline on the layout of this dissertation, and a brief summary on our contribution to the CHY formalism.

1.2 Outline of this work

This thesis is organized as follows. In Chapter 2, we review some introductory knowledge on the structure of gauge and gravity amplitudes. This review is not comprehensive, but emphasizes results that are useful in the subsequent chapters. Then we formally introduce the CHY formalism on tree level gauge and gravity in Chapter 3. The first two chapters are devoted to prepare the readers for understanding more technical studies presented next:[3]

- Chapter 4 is based on our paper [39]. It explores how to reproduce the known MHV Yang-Mills and gravity amplitudes (Parke-Taylor formula and Hodges formula).
- Chapter 5 is based on our paper [40], where we derived a new compact formula for single trace tree level MHV amplitudes for Einstein-Yang-Mills theory, using the CHY formalism.
- Chapter 6 is based on our paper [41], where we studied the correspondence between helicity configurations with the solutions to the scattering equation in four dimensions.

We then provide some possible future directions before we conclude in Chapter 7. Finally, in the three appendices, we give some important technical details that are used in our main text.

[3] A concise treatment can be found in [38].

CHAPTER 2

GAUGE AND GRAVITY AT TREE LEVEL

This chapter reviews and summarizes several well-known properties of tree level gauge and gravity amplitudes. The emphasis is on the gauge amplitude relations and the double copy between gauge and gravity. In Section 2.1, we introduce the spinor helicity formalism. In terms of spinor variables, gauge amplitudes are surprisingly simple. This is totally unexpected from what Feynman diagrams suggest. It was the invention of the spinor helicity formalism that gave birth to the field of modern scattering amplitude. In Section 2.2, we briefly review the traditional Feynman diagram approach to calculate the Yang-Mills amplitudes. The simplicity of the amplitudes and the existence of various amplitude relations clearly indicate that the gauge amplitudes have a very elegant mathematical structure. A lot of effort has been put on understanding this structure, and the pursuit, in turn, provides novel and more efficient ways to evaluate amplitudes. In principle, amplitudes of the perturbative gravity can also be obtained from the Feynman diagram approach, but as we show in Section 2.3, the complexity soon becomes unwieldy. String theory tells us that gravity amplitude is secretly an inner product of color ordered gauge amplitudes, and their expressions are well organized. Similarly, we show in Section 2.4 that the color kinematic duality is the key to ensure that the double copy of gauge amplitudes can lead to a meaningful gravity amplitude, even with certain class of matter interactions.

2.1 Spinor helicity formalism

First proposed by Xu, Zhang, and Chang [4] in 1984,[1] the spinor helicity formalism significantly simplifies the result of multigluon amplitudes, which earned its name "Chinese magic."

In 4d space-time, given a null vector p^μ, we have the identity:

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[1] The preprint came in 1984 while the paper was not published until 1987.

$$p^2 = \eta_{\mu\nu} p^\mu p^\nu = \det\left(p_\mu \sigma^\mu\right) = 0\,. \tag{2.1}$$

Consequently, the 2×2 matrix $p_{\alpha\dot{\alpha}} \equiv \left(p_\mu \sigma^\mu\right)_{\alpha\dot{\alpha}}$ must have co-rank one, such that we can decompose it into the direct product of two spinors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}} \equiv |p]\langle p|\,. \tag{2.2}$$

The left-handed spinor λ_α transforms as the $(1/2, 0)$ representation of $SL(2, \mathbb{C})$ while the right-handed one $\widetilde{\lambda}_{\dot{\alpha}}$ transforms as $(0, 1/2)$. Here $|p]$ and $\langle p|$ are the two short-hand notations for spinors. We can raise the index α and $\dot{\alpha}$ in the following way to form the dual spinors:

$$\epsilon^{\alpha\beta}\lambda_\beta = \lambda^\alpha \equiv [p|\,, \qquad\qquad \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\lambda}_{\dot{\beta}} = \widetilde{\lambda}^{\dot{\alpha}} \equiv |p\rangle\,. \tag{2.3}$$

If p^μ is a real momentum, then $p_{\alpha\dot{\alpha}}$ is an Hermitian matrix, which imposes that $\widetilde{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^*$. Once we complexify p^μ, which is a very useful technique in this field, then λ_α and $\widetilde{\lambda}_{\dot{\alpha}}$ become totally independent. If we manage to find one set of $\{\lambda_\alpha, \widetilde{\lambda}_{\dot{\alpha}}\}$ that makes Eq. (2.2) hold, the spinors rescaled by a complex number t:

$$\lambda_\alpha \;\to\; t^{-1}\lambda_\alpha, \qquad \widetilde{\lambda}_{\dot{\alpha}} \;\to\; t\widetilde{\lambda}_{\dot{\alpha}} \tag{2.4}$$

will also do the job. For real momentum, t is restricted to be a phase. We call this operation *little group rescaling*.[2] This point is important to make the degrees of freedom balanced in Eq. (2.2): now both sides have 3 real (complex) degrees of freedom, depending on whether p^μ is real or complex. By construction, $p_{\alpha\dot{\alpha}}$ is invariant under the little group rescaling.

2.1.1 Spinor identities

If we have two null momenta p^μ and q^μ, their inner product can be written in terms of spinor inner products as:

$$2p \cdot q \equiv 2p^\mu q_\mu = p_{\alpha\dot{\beta}} q^{\dot{\beta}\alpha} = -\langle pq\rangle[pq]\,, \tag{2.5}$$

where the angular and square brackets are the abbreviation of the spinor inner products:

$$\langle pq\rangle \equiv (\widetilde{\lambda}_p)_{\dot{\alpha}}(\widetilde{\lambda}_q)^{\dot{\alpha}}\,, \qquad [pq] \equiv (\lambda_p)^\alpha(\lambda_q)_\alpha\,, \tag{2.6}$$

[2] This redundancy is the little group associated with massless particles.

agreeing with the short-hand notations introduced in Eq. (2.2) and (2.3). The inner products are antisymmetric:

$$\langle pq\rangle = -\langle qp\rangle\,, \qquad [pq] = -[qp]\,. \tag{2.7}$$

In particular, we have the identity:

$$\langle pp\rangle = [pp] = 0\,, \tag{2.8}$$

both due to the antisymmetric nature of $\epsilon^{\alpha\beta}$. Similarly, we have the product $[p|\sigma^\mu|q\rangle$, which essentially states that the $(1/2, 1/2)$ representation of $SL(2,\mathbb{C})$ is the complex four-momentum representation of the Lorentz group. We can easily derive that:

$$[p|\sigma^\mu|q\rangle = \langle q|\overline{\sigma}^\mu|p]\,, \tag{2.9}$$

which follows directly from:

$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}} = (\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}\,. \tag{2.10}$$

If we further simplify our notation by using $|i] \equiv |p_i]$, etc, as we will do repeatedly in this dissertation, we can write down two important identities, which hold for arbitrary spinors, in a very nice form:

Fierz identity:

$$\langle 1|\overline{\sigma}^\mu|2]\langle 3|\overline{\sigma}_\mu|4] = -2\langle 13\rangle[24]\,, \qquad [1|\sigma^\mu|2\rangle[3|\sigma_\mu|4\rangle = -2[13]\langle 24\rangle\,. \tag{2.11}$$

This identity directly comes from the fact that:

$$(\overline{\sigma}_\mu)^{\dot{\alpha}\alpha}(\overline{\sigma}^\mu)^{\dot{\beta}\beta} = 2\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\,, \qquad (\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\,,$$

which is sometimes also called Fierz identity.

Schouten identity:

$$|1\rangle\langle 23\rangle + |2\rangle\langle 31\rangle + |3\rangle\langle 12\rangle = 0\,, \qquad |1][23] + |2][31] + |3][12] = 0\,. \tag{2.12}$$

It follows from the fact that the spinor space is two dimensional, such that the symmetrization of any three indices vanishes identically:

$$\epsilon_{\alpha(\beta}\epsilon_{\gamma\delta)} = 0\,, \qquad \epsilon_{\dot{\alpha}(\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta})} = 0\,.$$

If we have n scattering massless particles, there total momentum must be conserved:[3]

$$\sum_{i=1}^{n}(p_i)_{\alpha\dot{\alpha}} = 0\,. \tag{2.13}$$

Once we contract it with two arbitrary spinors $\langle j|$ and $|k]$, we get another identity, imposed by physics:

$$\sum_{i=1}^{n}\langle ji\rangle[ik] = 0\,. \tag{2.14}$$

This enables one to eliminate certain momentum spinors in a physical scattering process.

Occasionally, one needs to "cross" a particle from initial state to final state. This corresponds to the analytic continuation $p \to -p$. In this disseration, we use the convention:

$$|-p\rangle = -|p\rangle\,, \qquad |-p] = +|p]\,, \tag{2.15}$$

namely, we flip the sign of angular spinor.

Henceforth, we will use the angular and square spinor notation consistently, for the sake of simplicity. The index α and $\dot{\alpha}$ will be kept implicit whenever possible.

2.1.2 Polarization vector

The polarization ϵ_μ is a complex null vector that describes the helicity state of external gluon and graviton. In four dimensions, we only have two helicities for each particle with spin: positive and negative, the polarization of which is denoted as $\epsilon^\pm$. For a momentum p_i, the polarization $\epsilon_i^\pm$ must satisfy the transverse condition:

$$p_i \cdot \epsilon_i^\pm = 0\,. \tag{2.16}$$

According to Eq. (2.5), the transverse condition means that $\epsilon_i^\pm$ either shares $|i]$ or $\langle i|$ with $p_i = |i]\langle i|$. Therefore, we can write them as

$$\epsilon_i^+(\eta) \propto |i]\langle\eta|\,, \qquad \epsilon_i^-(\xi) \propto |\xi]\langle i|\,,$$

where $\langle\eta|$ and $|\xi]$ are two arbitrary reference spinors. They encode exactly the gauge freedom in the polarization vectors:

$$\epsilon_i \to \epsilon_i + C p_i\,, \qquad \forall C \in \mathbb{C}\,. \tag{2.17}$$

We can use the little group rescaling to fix the form of ϵ_i up to an overall constant depending on conventions. Gauge invariance of physical amplitudes requires $\langle\eta|$ and $|\xi]$ be

[3] It is customary to choose all particles be outgoing.

cancelled in final results, so that $\epsilon^{\pm}$ should be invariant under a rescaling of them. This prompts us to express:

$$\epsilon_i^+(\eta) = \sqrt{2} \times \frac{|i]\langle\eta|}{\langle\eta i\rangle}, \qquad \epsilon_i^-(\xi) = \sqrt{2} \times \frac{|\xi]\langle i|}{[\xi i]}, \tag{2.18}$$

where the coefficient $\sqrt{2}$ is just a convention. The vector form of $\epsilon^{\pm}$ is:

$$(\epsilon_i^+)_\mu = \frac{1}{\sqrt{2}}\frac{[i|\sigma_\mu|\eta\rangle}{\langle\eta i\rangle}, \qquad (\epsilon_i^-)_\mu = \frac{1}{\sqrt{2}}\frac{\langle i|\overline{\sigma}_\mu|\xi]}{[\xi i]}, \tag{2.19}$$

in which we simply use $\epsilon^{\pm}$, with the reference spinor dependence suppressed. Using the Fierz identity (2.11), we have the following results for inner products of polarization vectors:

$$\begin{aligned}
\epsilon_i^+ \cdot \epsilon_j^+ &= -\frac{[ij]\langle\eta_i\eta_j\rangle}{\langle i\eta_i\rangle\langle j\eta_j\rangle}, & \epsilon_i^- \cdot \epsilon_j^- &= -\frac{\langle ij\rangle[\xi_i\xi_j]}{[i\xi_i][j\xi_j]}, \\
\epsilon_i^+ \cdot \epsilon_j^- &= \frac{[i\xi_j]\langle j\eta_i\rangle}{\langle i\eta_i\rangle[j\xi_j]}, & \epsilon_i^- \cdot \epsilon_j^+ &= \frac{\langle i\eta_j\rangle[j\xi_i]}{[i\xi_i]\langle j\eta_j\rangle}.
\end{aligned} \tag{2.20}$$

Under the litte group rescaling (2.4), $\epsilon_i^{\pm}$ transforms as:

$$\epsilon_i^+ \;\to\; t_i^{-2}\epsilon_i^+, \qquad \epsilon_i^- \;\to\; t_i^{+2}\epsilon_i^-. \tag{2.21}$$

Gluons are spin one massless particles, and their helicity states are described by a single polarization vector $(\epsilon^{\pm})_\mu$. Gravitons are spin two massless particles, and their helicity states are described by a polarization tensor $(\epsilon^{\pm})_\mu(\tilde{\epsilon}^{\pm})_\nu$. The tilde over the second vector indicates that we can choose a different gauge from the first one.

2.2 Yang-Mills amplitudes

The Yang-Mills theory with the gauge group $U(N)$ is given by the Lagrangian:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\,\text{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right). \tag{2.22}$$

The field strength tensor $\boldsymbol{F}_{\mu\nu}$ equals:

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - \frac{ig}{\sqrt{2}}\left[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu\right], \tag{2.23}$$

where the gluon field $\boldsymbol{A}_\mu = A_\mu^a T^a$ is a Lorentz vector taking value in the Lie algebra $u(N)$. The gluons are thus in the adjoint representation, with a runs from 1 to $N^2 - 1$. As an

$U(N)$ gauge theory, we have one more particle, the "photon," associated with the identity matrix $\mathbb{1}$. The generators close the Lie algebra:

$$\left[T^a, T^b\right] = i\tilde{f}^{abc}T^c, \qquad a, b, c \in \{1, 2, \ldots, N^2 - 1\}. \tag{2.24}$$

Since all the generators commute with the photon generator $T^{N^2} = \mathbb{1}$, the photon does not interact with any gluon. Our structure constant $\tilde{f}^{abc}$ is related to the usual one through $\tilde{f}^{abc} = \sqrt{2}f^{abc}$, such that the generators are normalized as $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. This also accounts for the $\sqrt{2}$ in $\boldsymbol{F}_{\mu\nu}$. Similar to the spinor Fierz identity (2.11), we have the $U(N)$ Fierz identity for the $N \times N$ matrix (T^a): [4]

$$\sum_{a=1}^{N^2} (T^a)_j{}^{\bar{i}} (T^a)_l{}^{\bar{k}} = \delta_l{}^{\bar{i}} \delta_j{}^{\bar{k}}. \tag{2.25}$$

Instead of a comprehensive introduction, we only focus on the amplitudes of Yang-Mills. *Unless otherwise stated, all the amplitudes considered in this dissertation are at the tree-level.*

2.2.1 Color-ordered amplitudes

Color and kinematic factors are entangled in gluon n-point amplitudes calculated from traditional Feynman rules. As the first step towards revealing the unsung beauty in the amplitudes, we are going to strip off color factors and only study *color-ordered (partial) amplitudes* in the following.

Quantization of a gauge theory always start with the gauge fixing. We choose possibly the most amplitude-friendly gauge, *Gervais-Neveu gauge* [42]. First, we introduce the Lie algebra valued tensor:

$$\boldsymbol{H}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \frac{ig}{\sqrt{2}} \boldsymbol{A}_\mu \boldsymbol{A}_\nu. \tag{2.26}$$

The gauge choice $\boldsymbol{G}$ that leads to Gervais-Neveu gauge is $\boldsymbol{G} = \boldsymbol{H}^\mu{}_\mu$. The field component of $\boldsymbol{G}$ can be extracted by a trace:

$$G^a \equiv \mathrm{Tr}(\boldsymbol{G}T^a) = \partial^\mu A^a_\mu - \frac{ig}{\sqrt{2}} \mathrm{Tr}(\boldsymbol{A}_\mu \boldsymbol{A}^\mu T^a). \tag{2.27}$$

The path integral including this gauge fixing is

$$\mathcal{Z} = \exp\left[-\frac{i}{2}\int d^4x f^a f^a\right] \int (D\boldsymbol{A}) F(\boldsymbol{A}) \prod_{x,a} \delta(G^a - f^a) \exp\left(i \int d^4x \mathcal{L}_{\mathrm{YM}}\right). \tag{2.28}$$

[4] While $(\sigma^\mu)_{\alpha\dot{\alpha}}$ means that four vector is the bi-fundamental representation $(1/2, 1/2)$ of $SL(2, \mathbb{C})$, the adjoint representation $(T^a)_j{}^{\bar{i}}$ is the bi-fundamental representation $(N, \overline{N})$ of $U(N)$.

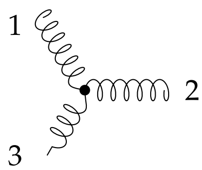
The Fadeev-Popov determinant $F(\boldsymbol{A})$ leads to a ghost $\boldsymbol{C}$, described by the Lagrangian:

$$\mathcal{L}_{\text{gh}} = \text{Tr}\left(\overline{\boldsymbol{C}} D_\mu D^\mu \boldsymbol{C} + \frac{ig}{\sqrt{2}} \overline{\boldsymbol{C}} \boldsymbol{C} \boldsymbol{H}^\mu{}_\mu\right), \qquad D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} \boldsymbol{A}_\mu,$$

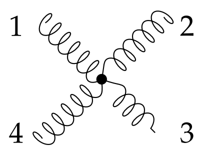
which can be ignored from now on since at tree-level ghosts do not appear. Actually, Gervais-Neveu gauge trades an ugly ghost Lagrangian for a very nice-looking gauge fixed Lagrangian:

$$\mathcal{L} = \text{Tr}\left(-\frac{1}{2}(\partial_\mu \boldsymbol{A}_\nu)(\partial^\mu \boldsymbol{A}^\nu) - i\sqrt{2}\, g(\partial_\mu \boldsymbol{A}_\nu)\boldsymbol{A}^\nu \boldsymbol{A}^\mu + \frac{g^2}{4} \boldsymbol{A}_\mu \boldsymbol{A}_\nu \boldsymbol{A}^\mu \boldsymbol{A}^\nu\right), \tag{2.29}$$

from which the Feynman rules are derived as:

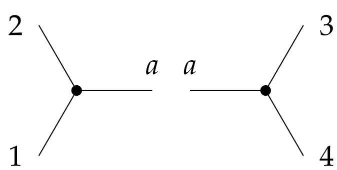


$$= g\sqrt{2}\, \text{Tr}(T^{a_1} T^{a_2} T^{a_3})(\eta^{\mu_1\mu_2} p_1^{\mu_3} + \eta^{\mu_2\mu_3} p_2^{\mu_1} + \eta^{\mu_3\mu_1} p_3^{\mu_2}) + (2 \leftrightarrow 3), \tag{2.30}$$



$$= g^2\, \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \times \eta^{\mu_1\mu_3} \eta^{\mu_2\mu_4} + (234) \text{ permutation}. \tag{2.31}$$

Notably, the color factors are encoded in a color trace. When two lower-point diagrams amalgamate into one higher-point diagram, we need to sum over the internal color index at the point they are glued. The $U(N)$ Fierz identity then makes the two color traces into one, for example,



$$= \sum_{a=1}^{N^2} \text{Tr}(T^{a_1} T^{a_2} T^{a})\, \text{Tr}(T^{a} T^{a_3} T^{a_4}) = \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}).$$

Therefore, the total n-particle tree-level amplitude $\mathcal{A}_n$ must be of the following form:

$$\mathcal{A}_n = g^{n-2}\, \text{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n}) A_n(12\ldots n) + (\text{ noncyclic perm. }), \tag{2.32}$$

where the summation is over the noncyclic permutation of $\{1, 2, \ldots, n\}$. The amplitude A_n associated to each color trace is called *color-ordered (partial) amplitude*. As a very commonly used shorthand notation, we use i to stands for the data $\{p_i, \epsilon_i\}$ of the particle i. *For simplicity, we will use solid line in the following to represent gluons in the Feynman diagrams, instead of wavy lines.*

The calculation of $\mathcal{A}_n$ is notorously difficult using the traditional Feynman diagram approach. One of the reasons is that even for moderate n, the number of Feynman diagrams involved quickly increases to one million [43], as shown in Table 2.1. As the first step towards simplification, Eq. (2.32) enables us to divide and conquer the total amplitude $\mathcal{A}_n$: we divide the Feynman diagrams into subsets, each of which contributes to only one partial amplitude A_n. For example, only the three diagrams in Figure 2.1 contribute to the partial amplitude $A_4(1234)$. The summation over permutation is not needed since other orders do not contribute to this partial amplitude. Such a division makes physical sense only if each partial amplitude is gauge invariant by itself. Namely, when replacing ϵ_i by p_i, not only the total amplitude $\mathcal{A}_n$, but also each partial amplitude A_n, satisfy the Ward identity:

$$A_n(\epsilon_i \to p_i) = 0\,. \tag{2.33}$$

This point is guaranteed by the *partial orthogonality* of the color traces:

$$\sum_{a_i=1}^{N^2} \mathrm{Tr}(T^{a_1}T^{a_2}\dots T^{a_n})\,[\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}})]^* = N^n\left[\delta_{\sigma I} + \mathcal{O}\left(\frac{1}{N^2}\right)\right], \tag{2.34}$$

where $\sigma \in S_n/\mathbb{Z}_n$ is a noncyclic permutation of $\{1,2,\dots,n\}$, and $\delta_{\sigma I} = 1$ if and only if σ is the identity element (up to a cyclic ordering). Although this is not a strict orthogonality, it is sufficient to ensure the gauge invariance of each color-ordered amplitude since those amplitudes A_n do not contain N while the gauge invariance must hold at each order of $\mathcal{O}(\frac{1}{N^2})$ expansion.

Table 2.1. Number of Feynman diagrams in gluon n-point amplitudes.

n	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

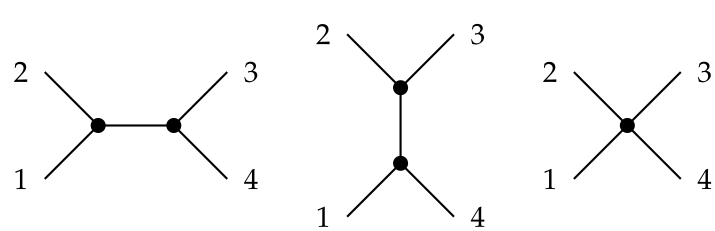


Figure 2.1. The color-ordered Feynman diagrams that contribute to $A_4(1234)$.

Before closing this subsection, we give a brief proof of Eq. (2.34). First, we notice that when the two T^a's are separated, Eq. (2.25) just rearranges the traces. If they are multiplied together, we gain a factor of N:

$$\sum_{a=1}^{N^2} (T^a)_j{}^{\bar{i}} (T^a)_i{}^{\bar{k}} = N \delta_j{}^{\bar{k}} . \tag{2.35}$$

This fact leads to the following proof:

proof of partial orthogonality: Since all the Lie algebra generators are Hermitian matrices, the complex conjugate on the trace just reverses its order:

$$\left[\mathrm{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\right)\right]^* = \mathrm{Tr}\left(T^{a_{\sigma(n)}} \ldots T^{a_{\sigma(2)}} T^{a_{\sigma(1)}}\right) .$$

Now we further simplify the notation by using

$$(a_1 a_2 \ldots a_n) \equiv \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n}) .$$

First, if we have $\sigma = I$, Eq. (2.34) becomes

$$\begin{aligned} \sum_{a_i} (a_1 a_2 \ldots a_{n-1} a_n)(a_n a_{n-1} \ldots a_2 a_1) &= \sum_{a_i} (a_1 a_2 \ldots a_{n-1} a_{n-1} \ldots a_2 a_1) \\ &= N \sum_{a_i} (a_1 a_2 \ldots a_{n-2} a_{n-2} \ldots a_2 a_1) \\ &= N^{n-2} \sum_{a_1} (a_1 a_1) = N^n . \end{aligned}$$

This accounts for the part N^n when $\sigma = I$. For $\sigma \neq I$, let us assume that after the first step, the second a_{n-1} is at another position:

$$\begin{aligned} \sum_{a_i} (a_1 a_2 \ldots a_{n-1} a_n)(a_n X \ldots Y a_{n-1} Z \ldots) &= \sum_{a_i} (a_1 a_2 \ldots a_{n-1} X \ldots Y a_{n-1} Z \ldots) \\ &= \sum_{a_i} (Z \ldots a_1 a_2 \ldots a_{n-2})(X \ldots Y) , \end{aligned}$$

namely, non-neighboring Fierz contraction does not contribute the factor N. Now the best hope is that

$$(Z \ldots a_1 a_2 \ldots a_{n-2})(X \ldots Y) = (a_1 a_2 \ldots a_{n-2})(a_{n-2} \ldots a_2 a_1)$$

such that we get N^{n-2} after the contraction. In this way, we have proved that the right hand side of Eq. (2.34) is

$$N^n \left[\delta_{\sigma I} + \mathcal{O}\left(\frac{1}{N^2}\right) \right] ,$$

which is the partial orthogonality. $\square$

2.2.2 Helicity classification

Before delving into calculations, we can get some valuable information from a simple dimensional analysis of the gauge amplitudes. The conclusion is that if there are only less than two particles having different helicities from the others, the amplitude must vanish. It also motivates us to classify amplitudes by the helicity configurations, which also represent the level of complexity in the amplitudes.

Our first task is to find out the mass dimension of $\mathcal{A}_n$ (also A_n) in four dimensions. The cross section for the 2 to $n-2$ scattering process $g_A g_B \to (n-2)g$ can be expressed formally as:

$$d\sigma \underbrace{|\mathbf{v}_A - \mathbf{v}_B|}_{\text{relative velocity}} = \underbrace{(E_A E_B)^{-1}}_{\text{initial energy}} \underbrace{\left[\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}}\right]^{n-2}}_{\text{phase space}} \underbrace{\delta^4(p_1 + \dots p_{n-2} - p_A - p_B)}_{\text{momentum conservation}} |\mathcal{A}_n|^2 \,. \tag{2.36}$$

The full expression can be found in any standard quantum field theory textbook, for example [1]. After writing down the mass dimension of each component, we can solve that of $\mathcal{A}_n$:

$$[m]^{-2} = [m]^{-2}[m]^{2n-4}[m]^{-4}[\mathcal{A}_n]^2 \implies [\mathcal{A}_n] = [A_n] = [m]^{4-n} \,. \tag{2.37}$$

An n-point tree-level Feynman diagram with only 3-point vertices has exactly $n-2$ vertices and $n-3$ internal propagators. Since the 3-point vertex has mass dimension one, we work out correctly that:

$$[A_n] = \frac{[m]^{n-2}}{[m]^{2n-6}} = [m]^{4-n} \,. \tag{2.38}$$

For each presence of 4-point vertices, we eliminate two adjacent 3-point vertices and the propagator in between so that the above counting still works. Now here comes the punchline: if we dot in n polarization vectors, there must be at least two of them get multiplied together, since there are at most $n-2$ momentum factor upstairs:

$$A_n \sim \sum \frac{(\epsilon_i \cdot \epsilon_j)(\epsilon_k \cdot p_l)^{n-2}}{(P_I^2)^{n-3}} + \sum \frac{(\epsilon_i \cdot \epsilon_j)^2(\epsilon_k \cdot p_l)^{n-4}(p_m \cdot p_s)}{(P_I^2)^{n-3}} + \cdots \,. \tag{2.39}$$

Now let us look at the case when all gluons have the same helicity. According to Eq. (2.20), if we choose all the reference spinors be the same:

$$|\eta_1\rangle = |\eta_2\rangle = \cdots = |\eta_n\rangle = |\eta\rangle \,, \qquad |\xi_1] = |\xi_2] = \cdots = |\xi_n] = |\xi] \,,$$

their inner products must vanish: $\epsilon_i^{\pm} \cdot \epsilon_j^{\pm} = 0$. Then the amplitudes must also vanish, according to Eq. (2.39):

$$A_n(1^+2^+ \dots n^+) = 0\,, \qquad A_n(1^-2^- \dots n^-) = 0\,. \tag{2.40}$$

Next, we flip one helicity, say particle 1, and choose the following reference spinors:

$$|\eta_2\rangle = \cdots = |\eta_n\rangle = |1\rangle\,, \qquad |\xi_2] = \cdots = |\xi_n] = |1]\,,$$

we can achieve $\epsilon_i^{\pm} \cdot \epsilon_j^{\pm} = 0$ and $\epsilon_1^{\mp} \cdot \epsilon_j^{\pm} = 0$, such that the following amplitudes also vanish:

$$A_n(1^-2^+3^+ \dots n^+) = 0\,, \qquad A_n(1^+2^-3^- \cdots n^-) = 0\,. \tag{2.41}$$

This luck ends if we flip one more particle. Suppose now we have particle 1 and 2 of negative helicity and all the others positive. With the reference spinors:

$$|\eta_1\rangle = |\eta_2\rangle = \eta\,, \qquad |\eta_3\rangle = |\eta_4\rangle = \cdots = |\eta_n\rangle = |1\rangle\,,$$

the only nonzero inner products are $\epsilon_2^- \cdot \epsilon_j^+$, so that the amplitude looks like:

$$A_n(1^-2^-3^+ \dots n^+) \sim \sum \frac{(\epsilon_2^- \cdot \epsilon_j^+)(\epsilon_k \cdot p_l)^{n-2}}{(P_I^2)^{n-3}}\,. \tag{2.42}$$

This amplitude is called maximally helicity violating (MHV) amplitude. Similarly, we can define N^{k}MHV amplitudes, which have $k+2$ negative helicity particles. The case $k = n-4$ is called anti-MHV, which has only two positive helicites. The anti-MHV amplitude can be obtained by a complex conjugation of the MHV.

2.2.2.1 Three-point amplitudes

We now show that all the 3-point Yang-Mills amplitudes can be determined without consulting Feynman diagrams. The 3-point special kinematics has one peculiarity that is not shared by any other higher points:

$$p_1 + p_2 + p_3 = 0 \implies (p_1 + p_2)^2 = -p_3^2 = 0\,,$$

where $p_{1,2,3}$ are *complex null* momenta. The above two relations together lead to

$$\langle 12\rangle[23] = 0\,, \qquad \langle 12\rangle[13] = 0\,, \qquad \langle 12\rangle[12] = 0\,,$$

namely, if $\langle 12\rangle \neq 0$, all the square brackets vanish: $[12] = [23] = [13] = 0$. If all momenta are real, then angular brackets also vanish since square and angular spinors are conjugate.

The conclusion is that *if we allow complex momenta, then under* 3*-point special kinematics, we can only have either nonzero angular brackets or square ones, not both.* Now let us look at amplitudes. First, we notice that when evaluating Feynman diagrams, polarization vector of each particle only appears in the numerator once per diagram. Thus under the rescaling (2.4), the amplitude must transform in the same way as the polarization vector:

$$A_n(\ldots\{t|i\rangle, t^{-1}|i], h_i\}\ldots) = t^{-2h_i} A_n(\ldots\{|i\rangle, |i], h_i\}\ldots)\,. \tag{2.43}$$

Using this property, we can determine the possible forms of 3-point gauge amplitudes. Suppose angular brackets are nonzero, on which the amplitude can only depends, then we must have:

$$A_3(1^{h_1}2^{h_2}3^{h_3}) \propto \langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}\,. \tag{2.44}$$

Next, we need to investigate the coefficient at each helicity configuration, modulo permutations:

$$A_3(1^-2^-3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \tag{2.45a}$$

$$A_3(1^+2^+3^-) = c_1\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle^3}\,, \tag{2.45b}$$

$$A_3(1^+2^+3^+) = c_2\frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,, \tag{2.45c}$$

$$A_3(1^-2^-3^-) = c_3\langle 12\rangle\langle 23\rangle\langle 31\rangle\,. \tag{2.45d}$$

Since A_3 has dimension $[m]$ in four dimensions, the coupling g in Eq. (2.45a) is thus dimensionless, which is exactly the one in Yang-Mills. In the real momentum limit, all angular brackets also vanish such that $A_3(1^-2^-3^+) = 0$, which is physical since real 3-point kinematics forbids this scattering. On the other hand, Eq. (2.45b) and (2.45c) blow up in the real limit, and the couplings have positive mass dimension: $[c_1] = [m]^2$, $[c_2] = [m]^4$. This is the typical behavior of nonlocal interactions, which is absent in Yang-Mills. Therefore, we require $c_1 = c_2 = 0$. Finally, in Eq. (2.45d) the coupling c_3 has negative mass dimension: $[c_3] = [m]^{-1}$, which corresponds to the nonrenormalizable F^3 interaction $\mathrm{Tr}(\boldsymbol{F}^\mu{}_\nu \boldsymbol{F}^\nu{}_\rho \boldsymbol{F}^\rho{}_\mu)$. Therefore, when angular brackets are nonzero, the only nonvanishing Yang-Mill 3-point amplitude in four dimensions is the MHV one (2.45a). The same analysis can be performed for the case of nonzero square bracket.

Our final result is that the only possibly nonzero 3-point amplitudes of four dimensional Yang-Mills are MHV and anti-MHV:[5]

$$A_3(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \qquad\qquad A_3(1^+2^+3^-) = \frac{[12]^3}{[23][31]}\,. \tag{2.46}$$

However, they cannot be nonzero at the same time. From the prospective of real kinematics, which one being nonzero depends on how we complexify the momenta: whether we deviate $|i\rangle$ or $|i]$ from their real kinematic values while momentum conservation still holds. To wrap up, we emphasize that to reach Eq. (2.46), we have only used: (1) rescaling property of massless spin one particles, (2) 3-point special kinematics, (3) locality and renormalizability. It is well known that in four dimensions the only local renormalizable quantum field theory for a massless spin one particle is Yang-Mills, and we do get uniquely the amplitude of what we want.

2.2.2.2 MHV amplitudes

The behavior of Eq. (2.46) actually persists to arbitrary n-point MHV and anti-MHV amplitude. They can be described by the famous Parke-Taylor formula [3]:

$$\begin{aligned} A_n(1^-2^-3^+\dots n^+) &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\dots\langle n1\rangle}\,, \\ A_n(1^+2^+3^-\dots n^-) &= \frac{[12]^4}{[12][23][34]\dots[n1]}\,. \end{aligned} \tag{2.47}$$

This behavior has been proved by Berends-Giele [10] and BCFW [44] recursive relation. Therefore, despite the sky-rocketing complexity in Feynman diagrams, the MHV amplitudes always turn out to be a single term at arbitrary n.

2.2.3 Gauge amplitude relations

The calculation of the color-ordered amplitude A_n is much simpler than the total amplitude $\mathcal{A}_n$, not only because it corresponds to fewer Feynman diagrams, but it also satisfies a number of nice relations, some are inherited from the color trace:

Cyclic symmetry:

$$A_n(123\dots n) = A_n(23\dots n1) = A_n(3\dots n12) = \cdots\,. \tag{2.48}$$

[5] The common factor g is neglected in all the following equations.

This property comes directly from the cyclic symmetry of the color traces, which ensures that in $A_n(123\ldots n)$ all the cyclic rotations of particles have been summed over. We can use symmetry to fix the position of particle 1, so that the total number of independent color-ordered amplitudes is reduced to $(n-1)!$.

Reflection:

$$A_n(123\ldots n) = (-1)^n A_n(n\ldots 321)\,. \tag{2.49}$$

For example, we can check that from Eq. (2.30),

$$\begin{aligned}
A_3(123) &= (\epsilon_1\cdot\epsilon_2)(p_1\cdot\epsilon_3) + (\epsilon_2\cdot\epsilon_3)(p_2\cdot\epsilon_1) + (\epsilon_3\cdot\epsilon_1)(p_3\cdot\epsilon_2)\,,\\
A_3(321) &= (\epsilon_3\cdot\epsilon_2)(p_3\cdot\epsilon_1) + (\epsilon_2\cdot\epsilon_1)(p_2\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_3)(p_1\cdot\epsilon_2)\\
&= -(\epsilon_1\cdot\epsilon_2)(p_1\cdot\epsilon_3) - (\epsilon_2\cdot\epsilon_3)(p_2\cdot\epsilon_1) - (\epsilon_3\cdot\epsilon_1)(p_3\cdot\epsilon_2)\,.
\end{aligned}$$

The general proof comes from dividing the total amplitude in the usual Feynman gauge, whose color factor is a chain of structre constants, into each color-ordered amplitude.

$U(1)$ **decoupling identity:**

$$A_n(123\ldots n) + A_n(213\ldots n) + A_n(231\ldots n) + \ldots + A_n(23\ldots 1n) = 0\,. \tag{2.50}$$

This identity follows from taking $T^{a_1} = \mathbb{1}$, namely, make the particle 1 be a photon. Since the photon does not interact with the other gluons, the total amplitude must vanish. Then all the amplitudes in Eq. (2.50) carry the color trace $\mathrm{Tr}(T^{a_2}T^{a_3}\ldots T^{a_n})$ so that they must add to zero due to the partial orthogonality.

All the above relations come with a careful analysis of the color trace structure. On the other hand, there exist other unexpected relations from the Feynman diagram perspective:

Kleiss-Kuijf (KK) relation [11]:

$$A_n(1,\boldsymbol{\alpha},n,\boldsymbol{\beta}) = (-1)^{|\boldsymbol{\beta}|}\sum_{\boldsymbol{\sigma}\in\mathrm{OP}(\boldsymbol{\alpha},\boldsymbol{\beta}^T)} A_n(1,\boldsymbol{\sigma},n)\,, \tag{2.51}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two collections of particles, $\boldsymbol{\beta}^T$ is the reverse ordering of $\boldsymbol{\beta}$, and $\mathrm{OP}(\boldsymbol{\alpha},\boldsymbol{\beta}^T)$ are those permutations of $\boldsymbol{\alpha}\cup\boldsymbol{\beta}^T$ that preserve the ordering within $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^T$. We can thus fix the positions of two particles, say, 1 and n, by the KK relation, so that the number of independent color-ordered amplitudes is reduced to $(n-2)!$. This relation can be derived from the properties of color trace.

Bern-Carrasco-Johansson (BCJ) relation [29]:

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i} p_2 \cdot p_j\right) A_n(1,\ldots,i,2,i+1,\ldots,n) = 0\,. \tag{2.52}$$

After we apply the $U(1)$ decoupling identity to $A_n(134\ldots n2)$, all the amplitudes are of the KK independent form $A_n(1,\boldsymbol{\sigma},n)$, so that Eq. (2.52) indeed is a new set of relations among those KK independent amplitudes. The number of independent color-ordered amplitudes is further reduced to $(n-3)!$. The existence of this relation is tied with the color kinematic duality, which will be discussed in Section 2.4

We now provide some examples to these highly nontrivial relations. In Eq. (2.51), if $|\boldsymbol{\beta}| = 1$, the KK relation reduces to the $U(1)$ decoupling identity (2.50). As a nontrivial 5-point example, we choose $\boldsymbol{\alpha} = \{2\}$ and $\boldsymbol{\beta} = \{4,3\}$, such that the KK relation leads to:

$$A_5(12543) = A_5(12345) + A_5(13245) + A_5(13425)\,. \tag{2.53}$$

Also at 5-point, the BCJ relation gives:

$$s_{23}A_5(13245) + (s_{23}+s_{24})A_5(13425) - s_{12}A_5(13452) = 0\,, \tag{2.54}$$

where $s_{ab} = 2p_a \cdot p_b$, and the momentum conservation has been used in the last term. After applying the $U(1)$ decoupling identity (2.50) to the last amplitude, we obtain a relation between the three KK independent amplitudes:

$$s_{12}A_5(12345) + (s_{12}+s_{23})A_5(13245) - s_{25}A_5(13425) = 0\,. \tag{2.55}$$

The existence of these relations and the simplicity of the Parke-Taylor formula strongly suggest that some elegant structures in the color-ordered amplitudes are obscured by the unnecessarily complicated Feynman diagram evaluation. The culprit is actually easy to spot: our insistence on off-shell gauge invariance. It introduces too many extra baggages that on-shell amplitudes do not depend on. Therefore, modern approach to amplitudes works with on-shell particles from the beginning.

2.3 Gravity amplitudes

The classical theory for gravity is the general relativity. If the matter field is given by the Lagrangian $\mathcal{L}_{\text{matter}}$, then the following *Einstein-Hilbert action* solves how the spacetime curves and how the matter propagates in it at the same time:

$$S_{\text{EH}} = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,(R+2\Lambda) + \int d^4x\sqrt{-g}\,\mathcal{L}_{\text{matter}}\,. \tag{2.56}$$

The definition of each symbol in S_{EH} is given by:

G_N	Newton's constant, we have $\kappa^2 \equiv 8\pi G_N = 1.69 \times 10^{-37}\text{GeV}^{-2}$ at $D = 4$
R	Ricci scalar, see below for its definition
Λ	cosmological constant, $\Lambda \sim (10^{-3}\text{eV})^4$ in our universe
g	the determinant of the metric $g_{\mu\nu}$

Quantum field theory and general relativity has long been plagued by various conventions in the literature, which constantly leads to a sign difference in expressions. In this dissertation, we use the most-negative signature for $g_{\mu\nu}$, and the following expressions for the geometric quantities involved:

$$
\begin{aligned}
&\text{Christoffel connection} \quad \Gamma^\lambda_{\mu\nu} : && \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \\
&\text{Riemann curvature tensor} \quad R^\lambda{}_{\sigma\mu\nu} : && R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma} \\
&\text{Ricci tensor} \quad R_{\mu\nu} : && R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \\
&\text{Ricci scalar} \quad R : && R = g^{\mu\nu} R_{\mu\nu} \,.
\end{aligned}
\tag{2.57}
$$

Our convention agrees with [45].

The equation of motion for $g_{\mu\nu}$, *the Einstein equation*, is obtained by

$$
\frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = 0 \,.
$$

First, by varying the metric, the matter action produces the *energy-momentum tensor:*

$$
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \,, \tag{2.58}
$$

which satisfies the conservation law $\nabla_\mu T^{\mu\nu} = 0$. With all the preparation, we are now ready to write down the Einstein equation:

$$
\frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = 0 \quad \Longrightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \,. \tag{2.59}
$$

The left hand side is usually called the *Einstein tensor:* $G_{\mu\nu} \equiv R_{\mu\nu} - (1/2) R g_{\mu\nu} - \Lambda g_{\mu\nu}$. First, we assume that the matter distribution is fixed, namely, $T_{\mu\nu}$ is given and Eq. (2.59) is used to solve $g_{\mu\nu}$. Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric tensors, so that there are 10 equations in Eq. (2.59) in four dimensions. However, $G_{\mu\nu}$ must satisfy 4 constraints $\nabla_\mu G^{\mu\nu} = 0$, so that there are only 6 independent equations in Eq. (2.59). At the first glance, the metric $g_{\mu\nu}$ should also have 10 degrees of freedom in four dimensions such that Eq. (2.59) could not solve them all. However, since two metrics related by a diffeomorphism are considered equivalent:

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\, g_{\rho\lambda}(x)\,, \tag{2.60}$$

and the diffeomorphism invariance is a gauge symmetry. Therefore, by a gauge fixing, we can eliminate 4 degrees of freedom in $g_{\mu\nu}$, so that the 6 independent equations in Eq. (2.59) is enough to solve $g_{\mu\nu}$. The *harmonic gauge* (or de Donder gauge):

$$\partial_\mu(g^{\mu\nu}\sqrt{-g}) = 0 \tag{2.61}$$

is very commonly used for perturbative gravity.

If we allow the matter field to interact with gravity, we need to associate the Einstein equation with the Euler-Lagrange equation of the matter field. For example, if we have a scalar field $\mathcal{L}_{\text{matter}} = g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$, we need to add in the equation of motion:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{dV(\phi)}{d\phi} = 0\,. \tag{2.62}$$

The physical meaning of these two equations are nicely summarized by John Wheeler:

Eq. (2.59):	matter tells spacetime how to curve
Eq. (2.62):	spacetime tells matter how to move

In this dissertation, we will study one such interaction, *the Einstein-Yang-Mills theory* (EYM), where $\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{YM}}$. According to Eq. (2.58), the energy momentum tensor of Yang-Mills reads:

$$T_{\mu\nu} = -\operatorname{Tr}(\boldsymbol{F}_{\mu\rho}\boldsymbol{F}_\nu{}^\rho) + \frac{1}{4}\, g_{\mu\nu}\operatorname{Tr}(\boldsymbol{F}_{\rho\sigma}\boldsymbol{F}^{\rho\sigma})\,, \tag{2.63}$$

through which gluons interact with gravitons. The definition of graviton in the weak field limit is the subject of the next subsection.

2.3.1 Perturbative gravity

In the weak field limit, we can expand the metric around the flat spacetime as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{2.64}$$

where $h_{\mu\nu}$ is the graviton field. Now we can treat $h_{\mu\nu}$ as a usual field on the flat background spacetime, just as a matter field. Therefore, the contraction of indices is made through the flat metric $\eta_{\mu\nu}$. Both the expansion of $g^{\mu\nu}$ (the inverse metric) and $\sqrt{-g}$ do not terminate:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\rho} h^{\rho\nu} + \mathcal{O}(h^3)\,,$$
$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{2}h_{\mu\nu}h^{\mu\nu} + \frac{1}{8}h^2 + \mathcal{O}(h^3)\,, \tag{2.65}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Consequently, the expansion generates an infinite series of interaction vertices that are cubic and higher in $h_{\mu\nu}$. As a local quantum field theory, the perturbative gravity is clearly nonrenormalizable, since we need to introduce an infinite number of counterterms. However, we can still use it as an effective field theory, valid at energy scale much lower than the Planck scale.

First, let us look at the pure gravity, with $\mathcal{L}_{\text{matter}} = 0$. It describes how gravitons propagate and scatter in an otherwise empty spacetime. Under the expansion (2.65), the first and second order of the Einstein-Hilbert Lagrangian are (with proper integral by parts and discarding boundary terms):

$$\begin{aligned} \sqrt{-g}\,R &= \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h & \cdots \mathcal{O}(h) \\ &\quad + \frac{1}{2}\left[\partial^\rho h^{\mu\nu}\partial_\mu h_{\rho\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu}\right] & \cdots \mathcal{O}(h^2) \end{aligned} \tag{2.66}$$

where $\partial^2 \equiv \partial_\mu\partial^\mu$. The $\mathcal{O}(h)$ term integrates to zero in the action since it is a total derivative. The $\mathcal{O}(h^2)$ terms are actually the free part of the weak field Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{EH}} &= -\frac{1}{2\kappa^2}\sqrt{-g}\,R \\ &= \frac{1}{4\kappa^2}\left(\partial_\mu h^{\mu\nu}\partial_\nu h - \partial^\rho h^{\mu\nu}\partial_\mu h_{\rho\nu} + \frac{1}{2}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu h\partial^\mu h\right) + \mathcal{O}(h^3)\,. \end{aligned} \tag{2.67}$$

Clearly, there is no mass term $h^{\mu\nu}h_{\mu\nu}$ in the Lagrangian, so that graviton is a massless particle. Varying the Lagrangian with respect to $\delta h^{\mu\nu}$ gives the equation of motion:

$$\frac{1}{2}\left(\partial^\rho\partial_\mu h_{\rho\nu} + \partial^\rho\partial_\nu h_{\rho\mu} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\partial^2 h\right) = 0\,, \tag{2.68}$$

where the left hand side is just the $\mathcal{O}(h)$ order of the Einstein tensor $G_{\mu\nu}$.

In Eq. (2.60), if we take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, an infinitesimal coordinate transformation $\widetilde{x}^\mu = x^\mu - \xi^\mu$ will lead to a gauge transformation in $h_{\mu\nu}$:

$$h_{\mu\nu} \;\rightarrow\; \widetilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{2.69}$$

Indeed, we can check that $\mathcal{L}_{\text{EH}}$ is invariant at $\mathcal{O}(h^2)$. Actually, it is the only possible combination quadratic in $h_{\mu\nu}$ that is gauge invariant [46]. The weak field expansion of the harmonic gauge is:

$$\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\right) = -\partial_\mu h^{\mu\nu} + \frac{1}{2}\partial^\nu h + \mathcal{O}(h^3)\,. \tag{2.70}$$

After this gauge fixing, the quadratic Lagrangian contains only two terms:

$$\mathcal{L}_{\mathrm{EH}} + \frac{1}{4\kappa^2}\left(\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h\right)^2 = \frac{1}{8\kappa^2}\left(\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu h \partial^\mu h\right), \tag{2.71}$$

which leads to the graviton propagator:

$$D_{\mu_1\nu_1\mu_2\nu_2}(p) = \frac{i}{2p^2}\left(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\right). \tag{2.72}$$

Like the Lorentz gauge $\partial_\mu A^\mu = 0$ in electrodynamics, there is left-over gauge freedom under the harmonic gauge. Actually, on-shell graviton only has two internal states: helicity ± 2 state described by the polarization tensor $(e^\pm)_{\mu\nu} \equiv (\epsilon^\pm)_\mu(\epsilon^\pm)_\nu$.

The interaction between graviton and matter can be obtained by expanding $\sqrt{-g}\mathcal{L}_{\mathrm{matter}}$ with respect to $h_{\mu\nu}$:

$$\mathcal{L}_{h\text{-matter}} = \sqrt{-g}\mathcal{L}_{\mathrm{matter}} = \frac{1}{2}T - \frac{1}{2}h^{\mu\nu}T_{\mu\nu} + \mathcal{O}(h^2), \tag{2.73}$$

where $T \equiv T_{\mu\nu}\eta^{\mu\nu}$ is the trace of the energy momentum tensor $T_{\mu\nu}$. Together with $\mathcal{L}_{\mathrm{EH}}$, we can reproduce the full Einstein equation at the weak field limit:

$$\partial^\rho\partial_\mu h_{\rho\nu} + \partial^\rho\partial_\nu h_{\rho\mu} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\partial^2 h = \kappa^2 T_{\mu\nu}. \tag{2.74}$$

For EYM, the leading order graviton-gluon interaction is 3-point, involving one graviton and two gluons.

The self-interaction between gravitons are described by $\mathcal{O}(h^3)$ and higher order terms in $\mathcal{L}_{\mathrm{EH}}$. Just from the index structure, we can tell that these terms are very complicated. For example, $\mathcal{O}(h^3)$ is given by all possible contractions between three $h_{\mu\nu}$ and two ∂_μ, with indices properly symmetrized. In all we have 171 separate terms at $\mathcal{O}(h^3)$, while this number grows to 2850 at $\mathcal{O}(h^4)$ [6]. Therefore, calculation of graviton amplitudes using traditional Feynman rule approach is very involved. For example, in calculating 4-point graviton amplitudes, the expressions of s, t and u amplitudes span one and a half page each, while the final result boils down to only one single term, like the Yang-Mills case [47]. Clearly, a new algorithm with complexity matching final results may better reveal the nature of perturbative gravity.

Despite the complexity in Feynman rules, the graviton 3-point amplitude can also be fixed through little group rescaling and locality. We can redefine the graviton field as $h_{\mu\nu} \to \kappa h_{\mu\nu}$, and write the action schematically as:

$$S_{\text{EH}} = \int d^4x \left(h\partial^2 h + \kappa h^2 \partial^2 h + \kappa^2 h^3 \partial^2 h + \ldots \right) . \tag{2.75}$$

The coupling constant of the 3-point vertex is κ, with mass dimension $[m]^{-1}$. Plugging $h_i = \pm 2$ into Eq. (2.44), we get:

$$\begin{aligned} M_3(1^-2^-3^+) &= \kappa \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}, & M_3(1^+2^+3^-) &= c_1 \frac{\langle 23\rangle^2\langle 31\rangle^2}{\langle 12\rangle^6}, \\ M_3(1^+2^+3^+) &= c_2 \frac{1}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}, & M_3(1^-2^-3^-) &= c_3 \langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 . \end{aligned} \tag{2.76}$$

Like the Yang-Mills case, the κ one is the correct graviton 3-point MHV amplitude, while $[c_1] = [m]^3$ and $[c_2] = [m]^7$ are two nonlocal interactions, absent in the gravity theory. However, the $[c_3] = [m]^{-5}$ one corresponds to the R^3 interaction, in which 3-point vertex has the form $(\partial^2 h)(\partial^2 h)(\partial^2 h)$. It appears in some modified gravity theory, but not in our Einstein gravity. Therefore, in four dimensions, the only nonzero 3-point graviton amplitudes are again MHV and anti-MHV:

$$M_3(1^-2^-3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}, \qquad M_3(1^+2^+3^-) = \frac{[12]^6}{[23]^2[31]^2} . \tag{2.77}$$

Remarkably, comparing Eq. (2.76) and Eq. (2.44), we get $M_3(123) = [A_3(123)]^2$, which holds not only between the Einstein gravity and Yang-Mills, but also between those non-minimal gravity and gauge theories with higher dimensional and nonlocal terms. Since the little group rescaling essentially means that for each Feynman diagram, the external wavefunction of each particle can only appear once in the numerator, it actually sorts out all possible field theoretical interactions. At three point, we have just derived the relation gravity = (gauge)^2. For higher point amplitudes, since we do not have any new coupling constants both in gravity and gauge, we hope this nice pattern to persist. Indeed it does in some less trivial form than a simple square. This will be explored our next two subsections.

2.3.2 Kawai-Lewellen-Tye (KLT) relation

Nonrenormalizable in nature with infinite interaction terms, the perturbative gravity resembles little with the much more organized Yang-Mills. However, gravity is secretly a double copy of gauge theory in some sense. The clue first comes in the study of string amplitudes. In 1986, Kawai, Lewellen, and Tye (KLT) showed that any n-point tree level closed string amplitude can be written as a sum over two n-point tree level open string amplitudes [7]. In the field theory limit (or technically "$\alpha' \to 0$ limit"), closed and open string amplitudes reduce to gravity and gauge amplitudes, respectively. Therefore, the simplest double-copy behavior shown in Eq. (2.77) does have a generalization to higher points.

2.3.2.1 Pure gravity

Using the modern amplitude language, we can write the generic KLT relation between gravity and Yang-Mills amplitudes in the form of an inner product over the $(n-3)!$ permutations of $\{2, 3, \ldots, n-2\}$, denoted as S_{n-3}:

$$M_n(12\ldots n) = -\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in S_{n-3}} A_n(1,\boldsymbol{\alpha},n,n-1) S[\boldsymbol{\alpha}|\boldsymbol{\beta}] A_n(1,\boldsymbol{\beta},n-1,n)\,, \tag{2.78}$$

where the KLT kernel $S[\boldsymbol{\alpha}|\boldsymbol{\beta}]$ is:

$$S[\boldsymbol{\alpha}|\boldsymbol{\beta}] = \prod_{i=2}^{n-2}\left[s_{\alpha_i 1} + \sum_{j=2}^{i-1} \theta(\alpha_j,\alpha_i)_{\boldsymbol{\beta}} s_{\alpha_j\alpha_i}\right] \tag{2.79}$$

We define the function θ to be:

$$\theta(\alpha_j,\alpha_i) = \begin{cases} 1 & \alpha_j \text{ is also before } \alpha_i \text{ in } \boldsymbol{\beta} \\ 0 & \text{otherwise} \end{cases}\,. \tag{2.80}$$

Note that α_j always comes before α_i in $\boldsymbol{\alpha}$. This formula is first written down and proved in [48], while we follow the notation of [33]. With this definition, $S[\boldsymbol{\alpha}|\boldsymbol{\beta}]$ is symmetric:[6]

$$S[\boldsymbol{\alpha}|\boldsymbol{\beta}] = S[\boldsymbol{\beta}|\boldsymbol{\alpha}]\,, \tag{2.81}$$

since the criterion in θ is symmetric in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: suppose we have $\alpha_i = \beta_k$, then

$$\sum_{j=2}^{i-1} \theta(\alpha_j,\alpha_i)_{\boldsymbol{\beta}} s_{\alpha_j\alpha_i} = \sum_{j=2}^{k-1} \theta(\beta_j,\beta_k)_{\boldsymbol{\alpha}} s_{\beta_j\beta_k}\,,$$

and Eq. (2.81) follows. At $n = 3$, we can immediately verify Eq. (2.77) from Eq. (2.78); at $n = 4$, we have $S[2|2] = s_{12}$ such that:

$$M_4(1234) = -A_4(1234) s_{12} A_4(1243)\,. \tag{2.82}$$

[6]Using the notation of the original paper [48], we have to reverse both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: $S_{\text{there}}[\boldsymbol{\alpha}|\boldsymbol{\beta}] = S_{\text{there}}[\boldsymbol{\beta}^T|\boldsymbol{\alpha}^T]$.

Now using the 4-point Parke-Taylor formula, we can get the 4-point graviton MHV amplitude:

$$\begin{aligned} M_4(1^-2^-3^+4^+) &= -A_4(1^-2^-3^+4^+)s_{12}A_4(1^-2^-4^+3^+) \\ &= -\frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} \\ &= -\frac{\langle 12\rangle^4[34]^4}{stu}\,, \end{aligned} \tag{2.83}$$

where the symbol s, t, and u are the usual Mandelstam variables:

$$s=(p_1+p_2)^2\,,\qquad t=(p_1+p_4)^2\,,\qquad u=(p_1+p_3)^2\,.$$

As a less trivial example, the $n=5$ KLT kernel consists of:

$$\begin{aligned} S[23|23] &= s_{12}(s_{13}+s_{23}) & S[23|32] &= s_{12}s_{13}\,, \\ S[32|23] &= s_{12}s_{13} & S[32|32] &= s_{13}(s_{12}+s_{32})\,, \end{aligned}$$

such that the KLT relation reads:

$$M_5(12345)=A_5(12345)s_{12}s_{34}A_5(14352)+A_5(13245)s_{13}s_{24}A_5(14253)\,, \tag{2.84}$$

in which the BCJ relation (2.52) has been used to simplify the result.

The KLT relation (2.78) can be understood in the following way. The Yang-Mills amplitude A_n is a summation over Feynman diagrams, each of which is multilinear in the polarizations ϵ_i. Thus we need exactly two A_n to match the polarization degrees of freedom of gravitons. The summation over different color ordering is essential since graviton amplitudes are symmetric under permutation while A_n is not. Finally, the coefficients s_{ij} come to cancel the double poles introduced by multiplying two gauge amplitudes. For example, in Eq. (2.82) the product $A_4(1234)A_4(1243)$ introduces a double pole at $(p_1+p_2)^2=0$, which is exactly canceled by the coefficient s_{12}. This is the reason why the KLT kernel $S[\boldsymbol{\alpha}|\boldsymbol{\beta}]$ soon becomes very complicated at higher points.

2.3.2.2 Einstein-Yang-Mills

Besides pure gravity, the total EYM amplitudes can also be constructed by the KLT product of amplitudes from Yang-Mills and a special double-colored scalar theory [49]:

$$\mathcal{L}=\mathrm{Tr}\left[\left(\nabla_\mu\boldsymbol{\Phi}^a\right)\left(\nabla^\mu\boldsymbol{\Phi}^a\right)\right]-\frac{i\kappa}{3!}f^{abc}\,\mathrm{Tr}\left([\boldsymbol{\Phi}^a,\boldsymbol{\Phi}^b]\boldsymbol{\Phi}^c\right)+\frac{g^2}{4}\,\mathrm{Tr}\left([\boldsymbol{\Phi}^a,\boldsymbol{\Phi}^b][\boldsymbol{\Phi}^a,\boldsymbol{\Phi}^b]\right) \tag{2.85}$$

where $\mathbf{\Phi}^a \equiv \phi^{a'a} T^{a'}$. The scalar field $\phi^{a'a}$ is in the adjoint representation of $U(N) \times U(N)$. However, only the *primed* color charges couple to our gluons. The n-point *total* EYM amplitudes with s gravitons and $r = n - s$ gluons can be expressed as:

$$\mathcal{M}_{n,r} = - \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in S_{n-3}} \widetilde{A}_{n,r}(\boldsymbol{\alpha}) S[\boldsymbol{\alpha}|\boldsymbol{\beta}] A_n(\boldsymbol{\beta}) \,, \tag{2.86}$$

where $A_n(\boldsymbol{\beta})$ is the n-point gluon color ordered amplitude complied to the permutation $\boldsymbol{\beta}$:

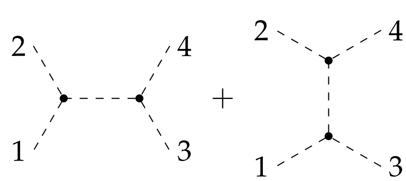
$$A_n(\boldsymbol{\beta}) \equiv A_n[g_1, \boldsymbol{\beta}(g_2, \ldots, g_{n-1}), g_{n-1}, g_n] \,.$$

As the other component of the KLT inner product, $\widetilde{A}_{n,r}$ is the n-point *partially color ordered amplitude* with r scalars and $n - r$ gluons, where the order of external legs are fixed according to $(1, \boldsymbol{\alpha}, n, n-1)$:

$$\widetilde{A}_{n,r}(\boldsymbol{\alpha}) \equiv \widetilde{A}_{n,r}[s_1^{a_1}, \boldsymbol{\alpha}(s_2^{a_2}, \ldots, s_r^{a_r}, g_{r+1}, \ldots, g_{n-2}), g_n, g_{n-1}] \,.$$

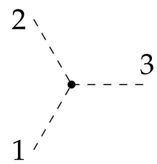
They are given by the Feynman rules derived from Eq. (2.85). The pure gluon part is the same as Eq. (2.30) and (2.31), while those involving scalars are shown in Figure 2.2 on the following page. We emphasize that $\widetilde{A}_{n,r}$ is color ordered in a', the color charge that couples to the gluons, while the uncouple color charge a is NOT color ordered. This is why we call $\widetilde{A}_{n,r}$ partially color ordered. If we require the resultant amplitudes to have only one single trace, then the external scalar legs should be connected. *We can understand Eq.* (2.86) *like this: if a gluon appears both in A_n and $\widetilde{A}_{n,r}$, it gets "squared" and promoted into a graviton. For the other gluons, they obtain color factors from the scalars and get summed over all possible color orderings.*

Now we give a few simple examples. First, we compute the 4-point pure scalar amplitude $\widetilde{A}_{4,4}(s_1^{a_1} s_2^{a_2} s_4^{a_4} s_3^{a_3})$, given by the following Feynman diagrams:

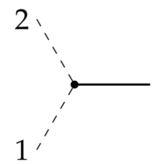


$$= -\frac{\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_4 a_3}}{s_{12}} - \frac{\tilde{f}^{a_2 a_4 b} \tilde{f}^{b a_3 a_1}}{s_{24}} \,. \tag{2.87}$$

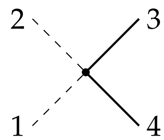
Then using $i\tilde{f}^{a_1 a_2 b} = \mathrm{Tr}([T^{a_1}, T^{a_2}] T^b)$, we have:



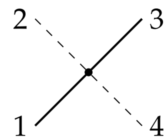
$= i\tilde{f}^{a_1 a_2 a_3}$



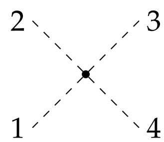
$= -\frac{i}{\sqrt{2}}\delta^{a_1 a_2}(p_1 - p_2)^\mu$



$= \frac{i}{2}\delta^{a_1 a_2}\eta^{\mu_3 \mu_4}$



$= -i\delta^{a_2 a_4}\eta^{\mu_1 \mu_3}$



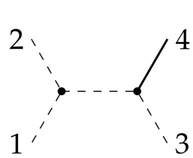
$= -i\delta^{a_1 a_3}\delta^{a_2 a_4} + \frac{i}{2}(\delta^{a_1 a_2}\delta^{a_3 a_4} + \delta^{a_1 a_4}\delta^{a_2 a_3})$

Figure 2.2. Feynman rules for partially color ordered amplitude $\widetilde{A}_{n,r}$.

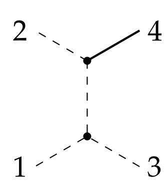
$$
\begin{aligned}
&- \widetilde{A}_{4,4}(s_1^{a_1} s_2^{a_2} s_4^{a_4} s_3^{a_3}) s_{12} A_4(g_1 g_2 g_3 g_4) \\
&= A_4(g_1 g_2 g_3 g_4)\Big[\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) + \frac{s_{23}}{s_{24}}\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\
&\qquad + \frac{s_{23}}{s_{24}}\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \frac{s_{12}}{s_{24}}\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) + \frac{s_{12}}{s_{24}}\mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\Big] \\
&= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4(g_1 g_2 g_3 g_4) + (234)\text{ permutations}\,, \qquad (2.88)
\end{aligned}
$$

namely, the final result is nothing but the total 4-point gluon amplitude $\mathcal{A}_4$. The last identity can be verified by expressing all the color ordered 4-point amplitudes in terms of $A_4(g_1 g_2 g_3 g_4)$, using KK and BCJ relation. We do not include the scalar 4-point vertex here since it contains a double trace structure. However, this vertex is crucial for constructing correct multitrace amplitudes through KLT. We will study such an example in Section 2.4.2.

To obtain a single trace EYM amplitude with a graviton, we can replace leg 4 by a gluon, for example, such that we now have $\widetilde{A}_{4,3}(s_1^{a_1}\, s_2^{a_2}\, g_4\, s_3^{a_3})$:



$+$



$$= i\tilde{f}^{a_1 a_2 a_3}\sqrt{2}\left(\frac{p_3 \cdot \epsilon_4}{s_{12}} - \frac{p_2 \cdot \epsilon_4}{s_{24}}\right). \qquad (2.89)$$

Suppose we choose gluon 4 be of negative helicity, with reference spinor $|2]$, we have:

$$\widetilde{A}_{4,3}(s_1^{a_1}\, s_2^{a_2}\, g_4^-\, s_3^{a_3}) = [\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) - \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2})]\,\frac{\langle 34\rangle[23]}{s_{23}[24]}\,,$$

such that the amplitude $\mathcal{M}_{4,3}(g_1^+ g_2^+ g_3^- h_4^-)$ can be calculated by:

$$\begin{aligned}\mathcal{M}_{4,3}(g_1^+ g_2^+ g_3^- h_4^-) &= -\widetilde{A}_{4,3}(s_1^{a_1} s_2^{a_2} g_4^- s_3^{a_3}) s_{12} A_4(g_1^+ g_2^+ g_3^- g_4^-) \\ &= [\mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3}) - \mathrm{Tr}(T^{a_1} T^{a_3} T^{a_2})] \times \frac{\langle 34 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \,. \end{aligned} \tag{2.90}$$

As a general pattern, the single trace MHV amplitudes of EYM carry a Parke-Taylor denominator ranging within the gluons.

The existence of such a KLT relation for EYM can be inferred from the heterotic string amplitudes. However, the list of functioning KLT pairs does not stop, and we require a better understanding on under what condition does this double copy trick work. A beautiful answer is given in Section 2.4 below.

2.3.3 Hodges formula for MHV gravity amplitudes

According to the KLT construction Eq. (2.78) and (2.86), we can see clearly that the helicity classification of pure gravity and EYM amplitudes is identical to that of the gauge theory, since they both contain one gauge amplitude in the KLT inner product.

The most compact formula for n-point tree level gravity amplitude is given by Hodges in 2012 [50, 51]:

$$M_n(- - + \ldots +) = \langle 12 \rangle^8 \bar{M}(12 \ldots n) \,, \tag{2.91}$$

where $\bar{M}$ is called the reduced gravity amplitude, which does not contain any information on helicity configurations. This quantity can be expressed by:

$$\bar{M}(12 \ldots n) = (-1)^{n+1} (-1)^{i+j+k+p+q+r} c_{ijk} c^{pqr} \det(\phi^{ijk}_{pqr}) \,, \tag{2.92}$$

where the c symbols are

$$c_{abc} = c^{abc} = \frac{1}{\langle ab \rangle \langle bc \rangle \langle ca \rangle} \,. \tag{2.93}$$

The $n \times n$ *Hodges matrix* ϕ is defined as:

$$\phi_{ab} = \frac{[ab]}{\langle ab \rangle} \quad (a \neq b) \,, \qquad \phi_{aa} = -\sum_{\substack{l=1 \\ l \neq a}}^{n} \frac{[al] \langle l\xi \rangle \langle l\eta \rangle}{\langle al \rangle \langle a\xi \rangle \langle a\eta \rangle} \,. \tag{2.94}$$

In the diagonal elements ϕ_{aa}, we have two reference spinors ξ and η. They represent a gauge freedom and the value of ϕ_{aa} does not depend on them: if we choose another spinor $\tilde{\eta}$, then we have:

$$\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\tilde{\eta}\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\tilde{\eta}\rangle}=\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\tilde{\eta}\rangle\langle a\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\tilde{\eta}\rangle\langle a\eta\rangle}$$
$$=\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle\eta\tilde{\eta}\rangle}{\langle a\xi\rangle\langle a\tilde{\eta}\rangle\langle a\eta\rangle}+\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\eta\rangle}$$
$$=\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\eta\rangle}\,, \tag{2.95}$$

where the first term in the second line vanishes due to the momentum conservation. Finally, ϕ^{ijk}_{pqr} is an $(n-3)\times(n-3)$ submatrix of ϕ with row $\{i,j,k\}$ and column $\{p,q,r\}$ deleted. It has been proved in [51] that ϕ has co-rank three and $\bar{M}$ is independent of the choice of $\{i,j,k\}$ and $\{p,q,r\}$. However, we are going to gain a better understanding on this point from the calculation of Cachazo-He-Yuan (CHY) formalism, to be discussed in detail in Chapter 4.

2.4 Color kinematic duality and double copy relation

In hindsight, the existence of KLT gives the first evidence on that gravity is secretly the square of a gauge theory. However, the KLT relation is still not transparent and powerful enough to fully reveal such a connection. Immediately, one can ask whether there is a way to "diagonalize" the KLT kernel such that gravity amplitudes are really some kind of square of gauge amplitudes. A deeper question one can ask is: what properties must the two component amplitudes satisfy such that their KLT inner product can produce a meaningful gravity theory? The best answer up to now is the *color-kinematic duality* [29], which is the main subject of this section.

2.4.1 Formal construction

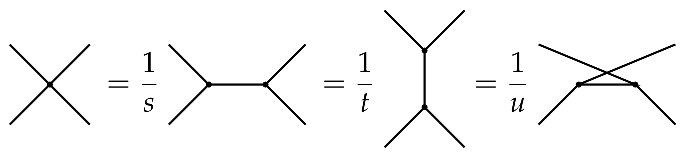
First, it is instructive to study the total gluon amplitude $\mathcal{A}_n$ instead of the color ordered one, since it respects the Bose symmetry, just as graviton amplitudes. Without losing generality, we can formally write the total amplitude as:

$$\mathcal{A}_n=\sum_{i\in\Gamma_3}\frac{c_i n_i}{\mathcal{P}_i^2}\,. \tag{2.96}$$

The summation is over all trivalent diagrams Γ_3 and $\mathcal{P}_i^2$ is the product of all internal propagators in the diagram i:

$$\mathcal{P}_i^2\equiv\prod_{\alpha_i}p_{\alpha_i}^2\,,\qquad \alpha_i\in\text{internal edges of diagram } i\,.$$

In the numerator, we can always factor out the color factor c_i from the kinematic part n_i. Here, c_i is a polynomial of the structure constant $\tilde{f}^{abc}$, while n_i is a Lorentz invariant quantity, composed of external momenta and polarizations.[7] Note that Γ_3 is not the set of Feynman diagrams. However, it is easy to reorganize Feynman diagrams into Γ_3. First, all trivalent Feynman diagrams belong to Γ_3. In the other diagrams, we can break all the 4-point vertices into two trivalent ones by inserting $1 = s/s = t/t = u/u$:



(2.97)

Which choice do we use depends on the color structure. In addition, the numerators n_i are not uniquely defined. This is not surprising since we allow a deformation of n_i by the gauge transformation $\epsilon_i \to \epsilon_i + \alpha p_i$, which does not change the total amplitude $\mathcal{A}_n$.

The numerators actually have a larger class of transformations that keep the total amplitude invariant. For example, the color factors in Figure 2.3 satisfy the Jacobi identity:

$$c_i + c_j + c_k \sim \tilde{f}^{ABe}\tilde{f}^{eCD} + \tilde{f}^{ADe}\tilde{f}^{eBC} + \tilde{f}^{ACe}\tilde{f}^{eDB} = 0\,. \tag{2.98}$$

In the denominator, the only propagator that is different in those three diagrams is s_i, s_j, and s_k highlighted by the thick gray lines. It is easy to check that with an arbitrary function Δ of kinematic variables, the following numerator transformation does not change the total amplitude:

$$n_i \;\to\; n_i + s_i\Delta \qquad\qquad n_j \;\to\; n_j + s_j\Delta \qquad\qquad n_k \;\to\; n_k + s_k\Delta\,, \tag{2.99}$$

due to the color Jacobi identity:

$$\delta\mathcal{A}_n \sim \Delta(c_i + c_j + c_k) = 0\,.$$

Eq. (2.99) is called *the dual gauge transformation.*

A natural question to ask is whether we can find a color basis in which the component (partial) amplitudes are invariant under the dual gauge transformation. Such a basis and the associated partial amplitudes can be constructed by the following way. First, we can

[7] In this context, it is more convenient to stick with the usual Feynman rules written in the Feynman gauge, where the color factor is given in terms of $\tilde{f}^{abc}$. They can be found in, for example, [1]. Without causing misunderstandings, we will simply call n_i numerator later.

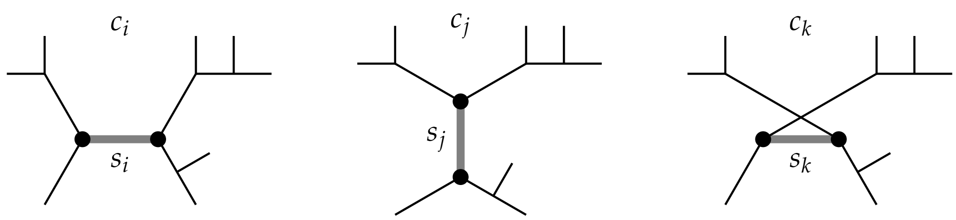
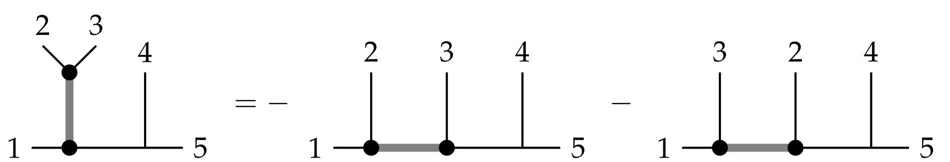
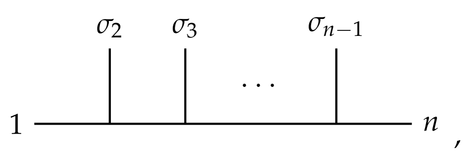


Figure 2.3. Three diagrams whose color factors satisfy the Jacobi identity.

pick out two external legs, say, 1 and n, and then land all the other external legs onto the edge connecting 1 and n by the color Jacobi identity. For example:



Then by this manipulation, we can transform each trivalent diagram into the following multiperipheral form:



(2.100)

whose color factor is

$$\tilde{f}^{a_1\sigma_2 b_2}\tilde{f}^{b_2\sigma_3 b_3}\tilde{f}^{b_3\sigma_4 b_4}\ldots\tilde{f}^{b_{n-2}\sigma_{n-1} n}\,.$$

Since any three multiperipheral diagrams cannot be related through a color Jacobi identity, we can use them as a basis, called Del Duca-Dixon-Maltoni (DDM) basis [52], to expand the total amplitude:

$$\mathcal{A}_n = (i)^{n-2}\sum_{\boldsymbol{\sigma}\in S_{n-2}}\tilde{f}^{a_1\sigma_2 b_2}\tilde{f}^{b_2\sigma_3 b_3}\tilde{f}^{b_3\sigma_4 b_4}\ldots\tilde{f}^{b_{n-2}\sigma_{n-1} n}A_n(1,\boldsymbol{\sigma},n)\,, \tag{2.101}$$

where $A_n(1,\boldsymbol{\sigma},n)$ are the usual color ordered amplitudes. This pursuit is in parallel with that in Section 2.2.1, where we define the color trace basis in which the color ordered amplitudes are invariant under the usual gauge transformation. The DDM basis demonstrates that there are $(n-2)!$ independent color factors under the Jacobi identity. This number agrees exactly with the $(n-2)!$ independent color ordered amplitudes after the particle 1 and n are fixed by cyclic and KK relations.

Now let us return to generic trivalent diagrams and discuss the properties of their numerators. We first make a rather bold claim and justify its legitimacy later: for each diagram, it is alway possible to find a numerator n_i such that it satisfies the same algebraic identities as the corresponding color factors:

$$c_i = -c_j \;\Leftrightarrow\; n_i = -n_j\,, \qquad c_i + c_j + c_k = 0 \;\Leftrightarrow\; n_i + n_j + n_k = 0\,. \tag{2.102}$$

This relation is the color-kinematic (C-K) duality [29]. First, we need to verify that the condition (2.102) is gauge invariant. Since each numerator is multilinear in polarizations, we can write $n_i = \epsilon_1^{\mu_1}(n_i)_{\mu_1}$. Since n_i is by definition an on-shell quantity, it does not contain any term X that makes $\epsilon_1^{\mu_1} X_{\mu_1} = 0$. As a result, the component $(n_i)_{\mu_1}$ must also satisfy Eq. (2.102). Under a gauge transformation $\epsilon_1 \to \epsilon_1 + \alpha p_1$, the numerators change as $n_i \to n_i + \Delta_i$, where $\Delta_i = \alpha p_1^{\mu_1}(n_i)_{\mu_1}$. Then it is easy to verify that:

$$n_i = -n_j \;\Rightarrow\; \Delta_i = -\Delta_j\,, \qquad n_i + n_j + n_k = 0 \;\Rightarrow\; \Delta_i + \Delta_j + \Delta_k = 0\,,$$

namely, the C-K duality is gauge invariant. For a set of n_i satisfying Eq. (2.102), we can perform Jacobi moves on the numerators and transform them into dual DDM basis, as what we have done to the color factors. According to Eq. (2.101), there are in all $(n-2)!$ C-K duality satisfying independent numerators $\hat{n}_i$ under Jacobi identities. We know that there are also $(n-2)!$ independent color ordered amplitudes $A_n^{(i)}$ under the cyclic and KK relations, such that there must be a linear relation in between:

$$A_n^{(i)} = \sum_{j=1}^{(n-2)!} \Theta^{ij} \hat{n}_j \qquad i = 1, 2, \ldots, (n-2)!\,. \tag{2.103}$$

Just from dimension analysis, we can tell that the entries of Θ^{ij} are solely composed of massless propagators. If the matrix Θ is invertible, then we would obtain a unique C-K duality satisfying basis $\hat{n}_i$ from any color ordered amplitudes, from which we can generate a unique set of C-K duality satisfying numerators by Jacobi identities. Actually, such numerators cannot be uniquely determined. It is always possible to add a set of higher dimensional operators to the Lagrangian in such a way that they add up to zero due to the Jacobi identity, but modify the numerators with the C-K duality preserved [53]. Therefore, if the C-K duality is true, there must exist one more kinematic linear relation between color ordered amplitudes such that Θ is degenerate. This relation is nothing but the famous BCJ relation (2.52):

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i} p_2\cdot p_j\right) A_n(1,\ldots,i,2,i+1,\ldots,n)=0\,. \tag{2.104}$$

More generally, the BCJ relation and the C-K duality are equivalent to each other: *if the amplitudes of a gauge theory (possibly interacting with matter fields) satisfy the BCJ relation, it signals the existence of the C-K duality.* Finally, we note that for Yang-Mills, the C-K duality satisfying numerators can indeed be constructed systematically at any n [54–58]. However, a closed and compact formula is still missing.

Now we arrive at the punchline of this story: if we have two gauge theories (possibly interacting with matter fields) whose color ordered amplitudes satisfy the BCJ relation, then we can construct a gravity amplitude (possibly interacting with matter fields) through the *double copy relation:*

$$M_n=\sum_{i\in\Gamma_3}\frac{n_i\widetilde{n}_i}{\mathcal{P}_i^2}\,. \tag{2.105}$$

The numerator n_i and $\widetilde{n}_i$ come from the gauge amplitudes:

$$\mathcal{A}_n^{(1)}=\sum_{i\in\Gamma_3}\frac{c_i n_i}{\mathcal{P}_i^2}\,,\qquad \mathcal{A}_n^{(2)}=\sum_{i\in\Gamma_3}\frac{c_i\widetilde{n}_i}{\mathcal{P}_i^2}\,, \tag{2.106}$$

in which one numerator, say n_i, should explicitly satisfy the C-K duality.[8] Again, it is instructive to prove that M_n constructed this way is gauge invariant.

Proof. Under $\epsilon\to\epsilon+\alpha p$, the numerator changes as $n_i\to n_i+\Delta_i$. Since the gauge amplitudes are invariant, we must have:

$$\sum_{i\in\Gamma_3}\frac{c_i\Delta_i}{\mathcal{P}_i^2}=0\qquad \sum_{i\in\Gamma_3}\frac{c_i\widetilde{\Delta}_i}{\mathcal{P}_i^2}=0\,.$$

Throughout our discussion, we have never specified the gauge group, such that the above two equations hold merely due to the Jacobi identity of the color factor. Therefore, we can

[8]The second numerator $\widetilde{n}_i$ does not need to satisfy the C-K duality explicitly, but it can be written in such a form due to the BCJ relation.

replace c_i by any other quantities that satisfy the same Jacobi identity, in particular, the C-K duality satisfying numerators:

$$\sum_{i\in\Gamma_3}\frac{n_i\widetilde{\Delta}_i}{\mathcal{P}_i^2}=0\qquad\sum_{i\in\Gamma_3}\frac{\Delta_i\widetilde{\Delta}_i}{\mathcal{P}_i^2}=0\,.$$

This immediately leads to:

$$\sum_{i\in\Gamma_3}\frac{(n_i+\Delta_i)(\widetilde{n}_i+\widetilde{\Delta}_i)}{\mathcal{P}_i^2}=\sum_{i\in\Gamma_3}\frac{n_i\widetilde{n}_i+n_i\widetilde{\Delta}_i+\Delta_i\widetilde{n}_i+\Delta_i\widetilde{\Delta}_i}{\mathcal{P}_i^2}=\sum_{i\in\Gamma_3}\frac{n_i\widetilde{n}_i+\Delta_i\widetilde{n}_i}{\mathcal{P}_i^2}\,. \tag{2.107}$$

Next, if $\widetilde{n}_i$ does not explicitly satisfy the C-K duality, there must exist another numerator $\widetilde{n}_i^{(2)}=\widetilde{n}_i+\Omega_i$ that does so. Since both numerators give the same gauge amplitude, we must have the identity:

$$\sum_{i\in\Gamma_3}\frac{c_i\Omega_i}{\mathcal{P}_i^2}=0\;\Rightarrow\;\sum_{i\in\Gamma_3}\frac{\Delta_i\Omega_i}{\mathcal{P}_i^2}=0\,,\qquad\sum_{i\in\Gamma_3}\frac{c_i\Delta_i}{\mathcal{P}_i^2}=0\;\Rightarrow\;\sum_{i\in\Gamma_3}\frac{\widetilde{n}_i^{(2)}\Delta_i}{\mathcal{P}_i^2}=0\,.$$

Therefore, we get:

$$\sum_{i\in\Gamma_3}\frac{\Delta_i\widetilde{n}_i}{\mathcal{P}_i^2}=\sum_{i\in\Gamma_3}\frac{\Delta_i\widetilde{n}_i^{(2)}-\Delta_i\Omega_i}{\mathcal{P}_i^2}=0\,, \tag{2.108}$$

and then the gauge invariance of M_n follows immediately. □

Comparing Eq. (2.105) with Eq. (2.96), we see that to get gravity from gauge amplitudes, we only need to replace the color factors c_i by the C-K duality satisfying n_i. Then following the same steps leading to Eq. (2.101), we can write the gravity amplitude in the dual DDM basis [53]:

$$M_n=\sum_{\boldsymbol{\sigma}\in S_{n-2}}n_{1|\sigma_2\sigma_3\ldots\sigma_{n-1}|n}A_n(1,\boldsymbol{\sigma},n)\,, \tag{2.109}$$

where each numerator factor $n_{1|\sigma_2\sigma_3\ldots\sigma_{n-1}|n}$ corresponds to one multiperipheral form in (2.100), and A_n is the gauge color ordered amplitude. The KLT relation (2.78) suggests a very natural arrangement:

$$n_{1|\boldsymbol{\alpha}|n-1,n}=-\sum_{\boldsymbol{\beta}\in S_{n-3}}S[\boldsymbol{\alpha}|\boldsymbol{\beta}]A_n(1,\boldsymbol{\beta},n,n-1)\,, \tag{2.110}$$

and $n_{1|\boldsymbol{\sigma}|n}=0$ if $\sigma_{n-1}\neq n-1$. For example, at $n=4$, we have:

$$n_{1|23|4}=-s_{12}A_4(1243)\qquad n_{1|32|4}=0\,.$$

At $n=5$ the only two nonzero factors are:

$$n_{1|234|5}=s_{12}s_{34}A_5(14352)\qquad n_{1|324|5}=s_{13}s_{24}A_5(14253)\,.$$

Then using Jacobi identities, we can reproduce a set of C-K duality satisfying numerators $\{n_i\}$. In this sense, the existence of C-K duality is equivalent to the KLT relation.

Less obviously, we can even transform Eq. (2.105) into the dual trace basis form [59]:

$$M_n = \sum_{\sigma \in S_{n-1}} \tau_{(1\sigma_2\sigma_3 \ldots \sigma_n)} A_n(1, \sigma_1, \sigma_2, \ldots, \sigma_n)\,, \tag{2.111}$$

where the dual trace $\tau_{(12\ldots n)}$ satisfies both the cyclic and KK relations, just as the usual color trace. The constructions of $\tau_{(12\ldots n)}$ can be found in [60–63].

To summarize, in this section we have shown that the scheme:

$$\text{gravity} = (\text{gauge})^2$$

is valid if and only if the gauge amplitudes satisfy the BCJ relation, although there is no rigorous proof yet in the mathematical sense. Furthermore, we have shown the equivalence:

$$\text{BCJ relation} \Leftrightarrow \text{C-K duality} \Leftrightarrow \text{double copy relation} \Leftrightarrow \text{KLT relation} \tag{2.112}$$

constructively at the tree level. However, the KLT relation is only valid at tree level, while the C-K duality and double copy relation can also be used to construct gravity loop integrands from the gauge theories ones [30].

2.4.2 Double copy examples

After a rather formal discussion, we provide several explicit 4-point calculations to manifest the relation (2.112). First, let us consider the total 4-point Yang-Mills amplitude, written in the color trace basis as:

$$\mathcal{A}_4 = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})A_4(1234) + (234 \text{ permutations}) \tag{2.113}$$

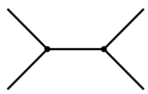
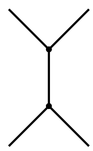
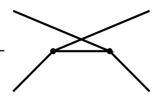
Using the KK relation, we can express all the partial amplitudes in terms of $A_4(1234)$ and $A_4(1324)$, and transform $\mathcal{A}_4$ into the DDM basis:

$$\mathcal{A}_4 = -\tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}A_4(1234) - \tilde{f}^{a_1a_3b}\tilde{f}^{ba_2a_4}A_4(1324)\,, \tag{2.114}$$

where we have used the identities:

$$
\begin{aligned}
c_s \equiv \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} &= -\operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\
&\quad + \operatorname{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) - \operatorname{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) \\
c_u \equiv \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_4 a_2} &= \operatorname{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) - \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\
&\quad - \operatorname{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \operatorname{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) \\
c_t \equiv \tilde{f}^{a_1 a_4 b} \tilde{f}^{b a_2 a_3} &= \operatorname{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) - \operatorname{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) \\
&\quad - \operatorname{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) + \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,.
\end{aligned}
\tag{2.115}
$$

We can verify that indeed $c_s + c_t + c_u = 0$. In the representation of (2.96), we have

$\mathcal{A}_4 =$  $+$  $+$  $= \dfrac{c_s n_s}{s} + \dfrac{c_t n_t}{t} + \dfrac{c_u n_u}{u}\,.$ (2.116)

Then using Eq. (2.115), we can express $A_4(1234)$ and $A_4(1324)$ in terms of the numerators:

$$
A_4(1234) = -\frac{n_s}{s} + \frac{n_t}{t} \qquad A_4(1324) = -\frac{n_t}{t} + \frac{n_u}{u}\,. \tag{2.117}
$$

Now if we impose the C-K duality $n_s + n_t + n_u = 0$ and choose the dual DDM basis as $(\hat{n}_1, \hat{n}_2) \equiv (n_s, n_t)$, we get

$$
\Theta = \begin{pmatrix} -\frac{1}{s} & \frac{1}{t} \\ -\frac{1}{u} & -\frac{1}{u} - \frac{1}{t} \end{pmatrix}, \tag{2.118}
$$

according to Eq. (2.103). It is easy to verify that indeed $\det(\Theta) = 0$, which indicates a linear relation within the two color ordered amplitudes:

$$
sA_4(1234) = uA_4(1324) = -n_s + \frac{sn_t}{t}\,. \tag{2.119}
$$

Following the double copy relation (2.105), the 4-point pure gravity amplitude should be:

$$
M_4(1234) = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{(n_s+n_t)^2}{u}\,. \tag{2.120}
$$

In this equation, if we replace n_s in terms of $A_4(1234)$ and n_t, according to Eq. (2.119), we find that in M_4, the n_t dependence is cancelled. What left is exactly:

$$
M_4(1234) = -\frac{st}{u}[A_4(1234)]^2 = -tA_4(1234)A_4(1324)\,, \tag{2.121}
$$

which is nothing but the KLT relation. Therefore, whether the double copy and KLT relation work hinges on the BCJ relation (2.119). At 4-point, the only nonzero amplitude is MHV, so it suffices to check it for the Parke-Taylor factor:

$$
\frac{s}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{-[12]\langle 24\rangle}{\langle 23\rangle\langle 34\rangle\langle 24\rangle\langle 41\rangle} = \frac{-[13]\langle 13\rangle}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle} = \frac{u}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\,.
$$

Indeed, the Yang-Mills amplitudes satisfy the BCJ relation, such that their double copy correctly reproduces a pure gravity amplitude.

As a second example, we consider the pure scalar 4-point amplitudes of the Lagrangian (2.85). The total 4-point amplitude has both the single trace and double trace contribution:

$$\widetilde{\mathcal{A}}_4 = \widetilde{\mathcal{A}}_4^{\text{s-t}} + \widetilde{\mathcal{A}}_4^{\text{d-t}} \,. \tag{2.122}$$

The single trace amplitude $\widetilde{\mathcal{A}}_4^{\text{s-t}}$ and the double trace amplitude $\widetilde{\mathcal{A}}_4^{\text{d-t}}$ have exactly the same color structure as the Yang-Mills ones:

$$\widetilde{\mathcal{A}}_4^{\text{s-t}} = \frac{c'_s \widetilde{n}_s^{\text{s-t}}}{s} + \frac{c'_s \widetilde{n}_t^{\text{s-t}}}{t} + \frac{c'_u \widetilde{n}_u^{\text{s-t}}}{u} \qquad \widetilde{\mathcal{A}}_4^{\text{d-t}} = \frac{c'_s \widetilde{n}_s^{\text{d-t}}}{s} + \frac{c'_s \widetilde{n}_t^{\text{d-t}}}{t} + \frac{c'_u \widetilde{n}_u^{\text{d-t}}}{u} \,, \tag{2.123}$$

where c'_s, c'_t, and c'_u are the same as those in Eq. (2.115), but with color index a'. Consequently, we have the color ordered amplitudes:

$$\begin{aligned}
\widetilde{A}_4^{\text{s-t}}(1234) &= -\frac{\widetilde{n}_s^{\text{s-t}}}{s} + \frac{\widetilde{n}_t^{\text{s-t}}}{t} & \qquad \widetilde{A}_4^{\text{s-t}}(1324) &= -\frac{\widetilde{n}_t^{\text{s-t}}}{t} + \frac{\widetilde{n}_u^{\text{s-t}}}{u} \\
\widetilde{A}_4^{\text{d-t}}(1234) &= -\frac{\widetilde{n}_s^{\text{d-t}}}{s} + \frac{\widetilde{n}_t^{\text{d-t}}}{t} & \qquad \widetilde{A}_4^{\text{d-t}}(1324) &= -\frac{\widetilde{n}_t^{\text{d-t}}}{t} + \frac{\widetilde{n}_u^{\text{d-t}}}{u} \,.
\end{aligned} \tag{2.124}$$

Then we can impose the C-K duality on the numerators such that using the double copy relation (2.105) with Yang-Mills, we can derive the single and double trace pure gluon 4-point EYM amplitudes in the KLT form:

$$\mathcal{A}_4^{\text{s-t}} = -t A_4(1234) \widetilde{A}_4^{\text{s-t}}(1324) \qquad \mathcal{A}_4^{\text{d-t}} = -t A_4(1234) \widetilde{A}_4^{\text{d-t}}(1324) \,. \tag{2.125}$$

Next, we verify whether the pure scalar amplitudes satisfy the BCJ relation. Otherwise the double copy would not give a meaningful gravity theory. The color ordered amplitude of the single trace part has already been calculated in Eq. (2.87):

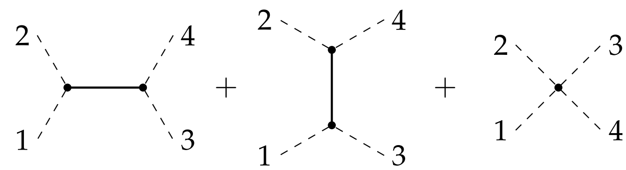
$$\begin{aligned}
\widetilde{A}_4^{\text{s-t}}(1234) &= -\frac{\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}}{s} + \frac{\tilde{f}^{a_1 a_4 b} \tilde{f}^{b a_2 a_3}}{t} \\
\widetilde{A}_4^{\text{s-t}}(1324) &= \frac{\tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_4 a_2}}{u} - \frac{\tilde{f}^{a_1 a_4 b} \tilde{f}^{b a_2 a_3}}{t} \,.
\end{aligned} \tag{2.126}$$

We can immediately see that they satisfy the BCJ relation:

$$s \widetilde{A}_4^{\text{s-t}}(1234) = u \widetilde{A}_4^{\text{s-t}}(1324) \,.$$

Actually, we have already checked in Eq. (2.88) that the KLT product correctly reproduces the total 4-point Yang-Mills amplitude as expected: $\mathcal{A}_4^{\text{s-t}} = \mathcal{A}_4$. The color ordered amplitude of the double trace part can be calculated from the Feynman rules in Figure 2.2:

$$\widetilde{A}_4^{\text{d-t}}(1234) =$$



$$= -\delta^{a_1 a_3}\delta^{a_2 a_4} - \delta^{a_1 a_2}\delta^{a_3 a_4}\left(\frac{u}{s}\right) - \delta^{a_1 a_4}\delta^{a_2 a_3}\left(\frac{u}{t}\right). \tag{2.127}$$

By exchanging the leg 2 and 3, we get:

$$\widetilde{A}_4^{\text{d-t}}(1324) = -\delta^{a_1 a_2}\delta^{a_3 a_4} - \delta^{a_1 a_3}\delta^{a_2 a_4}\left(\frac{s}{u}\right) - \delta^{a_1 a_4}\delta^{a_2 a_3}\left(\frac{s}{t}\right). \tag{2.128}$$

Again, it is easy to see that the BCJ relation is satisfied:

$$s\widetilde{A}_4^{\text{d-t}}(1234) = u\widetilde{A}_4^{\text{d-t}}(1324).$$

We note that if we forget the 4-point contact term in Figure 2.2, this BCJ relation will not hold.[9] The consequence is that all the multi-trace EYM amplitudes constructed from KLT would be wrong.

The KLT product (2.125) gives:

$$\mathcal{A}_4^{\text{d-t}} = \delta^{a_1 a_2}\delta^{a_3 a_4}[tA_4(1234)] + \delta^{a_1 a_3}\delta^{a_2 a_4}[tA_4(1324)] + \delta^{a_1 a_4}\delta^{a_2 a_3}[sA_4(1234)]. \tag{2.129}$$

The 4-point partial amplitude with the double trace structure $(a_1 a_2)(a_3 a_4)$ has the uniform expression:

$$A_4^{\text{d-t}}[(a_1 a_2)(a_3 a_4)|i^- j^-] = -\frac{s_{a_1 a_2}\langle ij\rangle^4}{\langle a_1 a_2\rangle\langle a_2 a_1\rangle\langle a_3 a_4\rangle\langle a_4 a_3\rangle}, \tag{2.130}$$

where the gluon i and j carry negative helicity. The generic n-point double trace pure gluon EYM amplitude is first written down in [36] using the Cachazo-He-Yuan (CHY) formalism, which is the main topic of Chapter 3.

[9]This happens in [49]. Using the Feynman rules therein, we can only derive the correct single trace EYM amplitudes.

2.5 Witten-RSV formalism

According to the work of Witten [17] and Roiban, Spardlin, Volovich [18], the superamplitude of the four dimensional $\mathcal{N} = 4$ super Yang-Mills can be written as:

$$\boldsymbol{A}_n = \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2 \Big[(\lambda_i)_\alpha - t_i \lambda_\alpha(\sigma_i) \Big] \times \prod_{m=0}^{d} \delta^2 \left[\sum_{i=1}^{n} t_i \sigma_i^m \widetilde{\lambda}_\alpha \right] \delta^4 \left(\sum_{i=1}^{n} t_i \sigma_i^m \eta_{iA} \right) . \tag{2.131}$$

The symbols in this formula have the following definitions:

- The two-spinor λ_i and $\widetilde{\lambda}_i$ are given by the null four momentum k_i through:

$$(k_i)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha (\widetilde{\lambda}_i)_{\dot{\alpha}} . \tag{2.132}$$

  The fermionic part of the momentum is given by η_{iA}, with $A = 1, 2, 3, 4$.

- The first delta function confines λ_i projectively to a degree d curve:

$$\lambda_\alpha(z) = \sum_{m=0}^{d} (\rho_m)_\alpha z^m . \tag{2.133}$$

- The degree d is connected to the N^{k}MHV sector through $d = n - k - 3$.
- The integral measure $d\mathcal{M}_{n,d}$ is defined as:

$$d\mathcal{M}_{n,d} = (d^{2d+2}\rho) \frac{d^n\sigma d^n t}{\text{Vol}[GL(2,\mathbb{C})]} \prod_{i=1}^{n} \frac{1}{t_i(\sigma_i - \sigma_{i+1})} \qquad (n+1 \equiv 1) . \tag{2.134}$$

In the last equation, We need to divide the volume of $GL(2,\mathbb{C})$ since the integrand is invariant under this world sheet symmetry.

The first two delta functions in Eq. (2.131) give the support of the superamplitude $\boldsymbol{A}_n$. The third fermionic delta function gives an expansion with respect to η_i, from which we can obtain all the component amplitudes, for example, pure gluon amplitudes $A_n(gg \dots g)$.

The support of the Witten-RSV delta function is:

$$\begin{aligned} &(\lambda_i)_\alpha - t_i \sum_{m=0}^{d} (\rho_m)_\alpha \sigma_i^m && (i = 1, 2 \dots n) \\ &\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} = 0 && (m = 0, 1 \dots d) . \end{aligned} \tag{2.135}$$

In all, there are $2n + 2d + 2$ equations. However, four of them are rather constraints interpretted as the momentum conservation:

$$\sum_{i=1}^{n} (\lambda_i)_\alpha (\widetilde{\lambda}_i)_{\dot{\alpha}} = \sum_{m=0}^{d} (\rho_m)_\alpha \sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} = 0 . \tag{2.136}$$

Therefore, there are only $2n+2d-2$ independent equations.[10] The unknown variables involved in Eq. (2.135) are $\{\sigma_i, t_i, (\rho_m)_\alpha\}$, whose number is $2n+2d+2$. Using the world sheet $GL(2,\mathbb{C})$ freedom, we can fix three σ's and one t, such that the total number of unknown variables is also $2n+2d-2$. Consequently, the solutions of Eq. (2.135) consist of only discrete points.

Interestingly, in the Witten-RSV integrand, only the *Parke-Taylor factor*, defined as:

$$PT(\boldsymbol{I}) = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \qquad (\sigma_{ij} \equiv \sigma_i - \sigma_j)\,, \tag{2.137}$$

where $\boldsymbol{I}$ stands for the identity permutation of $\{1,2,\ldots,n\}$, depends on the particle permutations. Thus, it should manifestly satisfy the KK and BCJ relation. Indeed, according to the calculation in Appendix B, the KK relation is satisfied by PT trivially as a complex number identity. However, PT satisfies the BCJ relation if and only if $\{\sigma\}$ solves the equation:

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_{ab}} = 0 \qquad a = 1,2\ldots n\,. \tag{2.138}$$

This is exactly the *scattering equation* later called by Cachazo, He, and Yuan [33–37]. In the following, we prove that the $\{\sigma\}$ solved from Eq. (2.135) must satisfy the scattering equation.

To start with, we propose a useful identity [18]:

$$p_m = \sum_{i=1}^{n} \frac{\sigma_i^m}{\prod_{j\neq i}(\sigma_i - \sigma_j)} = \begin{cases} 0 & m \leqslant n-2 \\ 1 & m = n-1 \\ \mathcal{O}(\sigma_i^{m-n+1}) & m \geqslant n \end{cases}\,. \tag{2.139}$$

Namely, p_m is a polynomial with leading terms:

$$\sum_{i=1}^{n} \sigma_i^{m-n+1}$$

when $m \geqslant n$, and it is constant otherwise. This identity follows directly from the global Cauchy theorem of multivariate complex analysis. Next, we define a degree $\widetilde{d} = n-d-2$ polynomial:

$$\widetilde{\lambda}_{\dot\alpha}(z) = \sum_{l=0}^{\widetilde{d}} (\widetilde{\rho}_l)_{\dot\alpha} z^l\,, \tag{2.140}$$

and another projection $\widetilde{t}_i$ such that $\widetilde{t}_i t_i = \prod_{j\neq i}\sigma_{ij}$.

---

[10] The conservation of supermomentum $\sum_i (\lambda_i)_\alpha \eta_{iA}$ can be similarly derived.

In hindsight, we come to evaluate the following integral:

$$I = \int d^{2\widetilde{d}+2}\widetilde{\rho} \prod_{i=1}^{n} \delta^2 \left[(\widetilde{\lambda}_i)_{\dot{\alpha}} - \widetilde{t}_i \widetilde{\lambda}_{\dot{\alpha}}(\sigma_i) \right], \tag{2.141}$$

which will eventually get related to the Witten-RSV integrand. We first perform a linear transformation to the variables of the delta function according to the $n \times n$ matrix

$$M_{mi} = t_i \sigma_i^m \qquad (m = 0, 1 \ldots n-1 \quad \text{and} \quad i = 1, 2 \ldots n). \tag{2.142}$$

The result is:

$$\begin{aligned} I &= \int d^{2\widetilde{d}+2}\widetilde{\rho}\ [\det(M)]^2 \prod_{m=0}^{n-1} \delta^2 \left[\sum_{i=1}^{n} M_{mi} (\widetilde{\lambda}_i)_{\dot{\alpha}} - \sum_{i=1}^{n} M_{mi} \widetilde{t}_i \widetilde{\lambda}_{\dot{\alpha}}(\sigma_i) \right] \\ &= \left[V \prod_{i=1}^{n} t_i \right]^2 \int d^{2\widetilde{d}+2}\widetilde{\rho} \prod_{m=1}^{n-1} \delta^2 \left[\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} - \sum_{l=0}^{\widetilde{d}} (\widetilde{\rho}_l)_{\dot{\alpha}} p_{l+m} \right], \end{aligned} \tag{2.143}$$

where V is the Vandermonde determinant:

$$V = \prod_{1 \leqslant i < j \leqslant n} (\sigma_i - \sigma_j).$$

Inside the delta functions, since the largest value of l is $\widetilde{d}$, the second term is identically zero when $0 \leqslant m \leqslant n - \widetilde{d} - 2 = d$, such that

$$\begin{aligned} I = \left[V \prod_{i=1}^{n} t_i \right]^2 & \prod_{m=0}^{d} \delta^2 \left(\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} \right) \\ & \times \int d^{2\widetilde{d}+2}\widetilde{\rho} \prod_{m=d+1}^{n-1} \delta^2 \left[\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} - \sum_{l=0}^{\widetilde{d}} (\widetilde{\rho}_l)_{\dot{\alpha}} p_{l+m} \right]. \end{aligned} \tag{2.144}$$

The integral in the second line is trivial:

$$\int d^{2\widetilde{d}+2}\widetilde{\rho} \prod_{m=d+1}^{n-1} \delta^2 \left[\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} - \sum_{l=0}^{\widetilde{d}} (\widetilde{\rho}_l)_{\dot{\alpha}} p_{l+m} \right] = 1,$$

since we have $\det(p_{l+m}) = 1$. The reason is that the $d \times d$ matrix p_{l+m} is upper triangular, with all diagonal elements being one. This derivation establishes the relation:

$$\prod_{m=0}^{d} \delta^2 \left(\sum_{i=1}^{n} t_i \sigma_i^m (\widetilde{\lambda}_i)_{\dot{\alpha}} \right) = \left[V \prod_{i=1}^{n} t_i \right]^{-2} \int d^{2\widetilde{d}+2}\widetilde{\rho} \prod_{i=1}^{n} \delta^2 \left[(\widetilde{\lambda}_i)_{\dot{\alpha}} - \widetilde{t}_i \widetilde{\lambda}_{\dot{\alpha}}(\sigma_i) \right], \tag{2.145}$$

such that the second equation of Eq. (2.135) is equivalent to

$$(\widetilde{\lambda}_i)_{\dot{\alpha}} - \widetilde{t}_i \widetilde{\lambda}_{\dot{\alpha}}(\sigma_i) = 0 \qquad (i = 1, 2 \ldots n). \tag{2.146}$$

After multiplying it with the first equation of Eq. (2.135), we get:

$$(\lambda_i)_\alpha(\widetilde{\lambda}_i)_{\dot{\alpha}} = \frac{\lambda_\alpha(\sigma_i)\widetilde{\lambda}_{\dot{\alpha}}(\sigma_i)}{\prod_{j\neq i}(\sigma_i - \sigma_j)} \qquad i = 1, 2 \ldots n \,, \tag{2.147}$$

which is nothing but the four dimensional scattering equation [32]. The relation with the canonical form (2.138) is derived in Appendix C. However, Eq. (2.138) is more general than this equation since Eq. (2.138) is valid in arbitrary dimensions.

CHAPTER 3

CACHAZO-HE-YUAN FORMALISM

Starting from 2013, Cachazo, He, and Yuan published a series of papers [33–37], proposing a unified recipe for calculating n-point tree level massless field theory amplitude in arbitrary spacetime dimensions:[1]

$$\int \frac{dz_1 \ldots dz_n}{\text{Vol}\left[SL(2,\mathbb{C})\right]} \left[\prod_{a=1}^{n}{}' \delta\left(f_a\right) \mathcal{I}_n \right] . \tag{3.1}$$

It is now called *the CHY formalism.* The meaning of each building block in this equation is listed below:

Symbol	Meaning	Note
$\mathcal{I}_n$	the CHY integrand	specifies the theory see Section 3.3
f_a	the scattering equation	see Eq. (3.5)
$\prod_{a=1}^{n}{}' \delta\left(f_a\right)$	the permutation invariant delta function	see Eq. (3.20)
$\frac{dz_1 \ldots dz_n}{\text{Vol}\left[SL(2,\mathbb{C})\right]}$	integration measure	see Eq. (3.24)

They will be discussed in detail later in this chapter. Currently, the CHY formalism exists for the following theories: Yang-Mills, pure gravity, Einstein-Yang-Mills, Yang-Mills-scalar, ϕ^3, ϕ^4, Dirac-Born-Infeld, nonlinear sigma model, and special Galileon [37], F^3 and R^3 theories [64]. All of them are formulated in a way that is valid in arbitrary spacetime dimensions.

The scattering equations $f_a = 0$ depend only on the external kinematic data, whose solutions provide the support for all physical amplitudes. Then the integrand $\mathcal{I}_n$ specifies the dynamics of different theories. Moreover, the requirement of world sheet $SL(2,\mathbb{C})$ invariance of the form:

[1]The integration should be understood as a contour integral in the n-punctured moduli space of the Riemann sphere.

$$(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n) \prod_{a=1}^{n}{}' \delta(f_a)\, \mathcal{I}_n \tag{3.2}$$

significantly constrains the possible forms of the integrand. In this sense, building amplitudes is like the game of mix-and-match guided by a symmetry principle. Many unexpected elegant structures have been discovered in this way in a large class of theories. The CHY formalism is thus a firm step towards conquering the territory beyond four dimensional Yang-Mills and gravity, and their supersymmetrization.

3.1 The scattering equations

Long before the scattering equation was rediscovered and endowed new physical importance by Cachazo, He, and Yuan, it has already made the appearance in the early days of string theory [65] (for a review, see [66]). The original motivation is to find a null map $P^\mu(z)$ from the world sheet to the lightcone, such that the momenta of physical massless particles are associated to the punctures on the world sheet. For n scattering particles with momentum conservation

$$k_1 + k_2 + \ldots + k_n = 0\,, \qquad k_a^2 = 0\,.$$

we require that each k_a^μ is the residue of $P^\mu(z)$ at a puncture σ_a:[2]

$$k_a^\mu = \oint_{|z-\sigma_a|=\epsilon} \frac{dz}{2\pi i}\, P^\mu(z)\,.$$

The map $P^\mu(z)$ can thus be written as:

$$P^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a}\,. \tag{3.3}$$

Furthermore, the null condition we have imposed on $P^\mu(z)$ requires that:

$$P^2(z) = \sum_{a=1}^{n}\sum_{b=1}^{n} \frac{k_a \cdot k_b}{(z-\sigma_a)(z-\sigma_b)} = 0\,. \tag{3.4}$$

As a holomorphic function in z, it is equivalent to imposing the zero residue at each puncture σ_a, which leads to the *scattering equation* (SE):

$$f_a \equiv \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \qquad a \in \{1, 2, \ldots, n\}\,. \tag{3.5}$$

---

[2]For tree level scattering, the world sheet is a Riemann sphere, or $\mathbb{C}P^1$. At loop level, the world sheet would be some higher genus Riemann surface, for example, torus for one loop. Consequently, the scattering equation will take another form, which will not be discussed in this dissertation.

In 1987, Gross and Mende [67] found that high energy string amplitudes are dominated by the saddle point of the Koba-Nielsen exponent, determined just by the SE (3.5). The high-energy limit can be effectively taken by setting $\alpha' \to \infty$, while the usual field theory emerges at low energy with $\alpha' \to 0$. To answer the question why the SE, as an object derived in the opposite limit, should have anything to do with the field theory amplitudes, people have developed new types of string theories that possibly have the CHY formula as the field theory limit, like the ambitwistor string [68–70] and the chiral string [71, 72].

3.1.1 Properties of the scattering equation

In this subsection, we investigate some crucial properties of the solutions to the SE. First, the SE is invariant under the world sheet $SL(2,\mathbb{C})$ transformation in the sense that given a solution $\{\sigma_a\}$, the following set $\{\zeta_a\}$

$$\left\{ \zeta_a = \frac{a\sigma_a + b}{c\sigma_a + d} \middle| \, a,b,c,d \in \mathbb{C},\ ad - bc = 1 \right\}$$

is also a solution. Indeed, it is easy to verify that:

$$\sum_{b\neq a} \frac{s_{ab}}{\zeta_a - \zeta_b} = (c\sigma_a + d) \sum_{b\neq a} \frac{(c\sigma_b + d)s_{ab}}{\sigma_a - \sigma_b} = (c\sigma_a + d)^2 \sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0\,.$$

To obtain the second equality, we have used the fact that:

$$c\sigma_b + d = c(\sigma_b - \sigma_a) + (c\sigma_a + d)\,, \qquad \sum_{b\neq a} s_{ab} = -k_a^2 = 0\,.$$

The consequence of this $SL(2,\mathbb{C})$ invariance is that we can "fix the gauge" by specifying the positions of any three punctures. For example, a very convenient choice is

$$\sigma_{n-2} = 0 \qquad \sigma_{n-1} = 1 \qquad \sigma_n = \infty\,.$$

Now that the SE (3.5) consists of n polynomial equations, it seems that the SE were overdetermined and should not have any solution. Actually, there are three linear relations between the set $\{f_a\}$:

$$P(f) \equiv \sum_{a=1}^{n} f_a = 0 \qquad Q(f) \equiv \sum_{a=1}^{n} \sigma_a f_a = 0 \qquad R(f) \equiv \sum_{a=1}^{n} \sigma_a^2 f_a = 0\,, \tag{3.6}$$

such that there are only $n-3$ independent equations in Eq. (3.5), agreeing with the number of variables.

proof of Eq. (3.6). The first relation in Eq. (3.6):

$$\sum_{a=1}^{n} f_a = \sum_{a=1}^{n}\sum_{b\neq a}\frac{s_{ab}}{\sigma_a-\sigma_b} = 0$$

comes from the fact that $s_{ab}/(\sigma_a-\sigma_b)$ is antisymmetric while the summation $\sum_{a=1}^{n}\sum_{b\neq a}$ is symmetric. The second relation $Q(f)=0$ can be proved as:

$$\sum_{a=1}^{n}\sigma_a f_a = \sum_{a=1}^{n}\sum_{b\neq a}\frac{\sigma_a s_{ab}}{\sigma_a-\sigma_b} = \sum_{a=1}^{n}\sum_{b\neq a}\frac{\sigma_b s_{ab}}{\sigma_a-\sigma_b} = -\sum_{a=1}^{n}\sum_{b\neq a}\sigma_a f_a = 0\,.$$

To obtain the second equality, we need to use the identity $\sigma_a = (\sigma_a-\sigma_b)+\sigma_b$ and the momentum conservation. Then we exchange the dummy index a and b to reach the third equality. For the third relation $R(f)=0$, we have

$$\sum_{a=1}^{n}\sigma_a^2 f_a = \sum_{a=1}^{n}\sum_{b\neq a}\frac{\sigma_a^2 s_{ab}}{\sigma_a-\sigma_b} = \sum_{a=1}^{n}\sum_{b\neq a}\frac{-\sigma_b^2 s_{ab}}{\sigma_a-\sigma_b} = \sum_{a=1}^{n}\sum_{a\neq b}\frac{(\sigma_a^2-\sigma_b^2)s_{ab}}{2(\sigma_a-\sigma_b)} = \sum_{a=1}^{n}\sum_{b\neq a}\frac{(\sigma_a+\sigma_b)s_{ab}}{2}\,.$$

Again we have used the renaming trick to derive the second equality. This result is identically zero since

$$\frac{1}{2}\sum_{a=1}^{n}\sum_{b\neq a}(\sigma_a+\sigma_b)s_{ab} = \sum_{a=1}^{n}\sigma_a\sum_{b\neq a}s_{ab} = 0\,,$$

where we have renamed b to a in the second term. □

The SE has a very simple and compact form, as one can tell from Eq. (3.5), but it is very difficult to solve. Actually, there are in all $(n-3)!$ solutions. The first derivation is given in [33], using a induction at the soft limit. To start the induction, we first solve the SE at $n=4$. It is easy to show that all the four equations gives the same solution:

$$\sigma_1 = -\frac{s_{12}}{s_{23}}$$

if we fix the gauge as $\{\sigma_2,\sigma_3,\sigma_4\} = \{0,1,\infty\}$. Thus at $n=4$ there is only one solution. Now suppose we have $n-1$ particles satisfying the momentum conservation $k_1+k_2+\ldots+k_{n-1}=0$, we can write down and solve the SE with $n-1$ particles:

$$f_a = \sum_{b=1 b\neq a}^{n-1}\frac{s_{ab}}{\sigma_a-\sigma_b} = 0, \qquad a\in\{1,2,\ldots,n-1\}\,. \tag{3.7}$$

We assume that this set of equations has $(n-4)!$ solutions, as the inductive assumption. Now the punchline is that we only add the particle n softly as:

$$s_{nb} = \epsilon\hat{s}_{nb} \qquad \epsilon\to 0\,,$$

such that the momentum conservation of the previous $n-1$ particles is preserved at the zeroth order of ϵ. Consequently, the previous $(n-4)!$ solutions of $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are also unchanged. With this new particle, we get one more equation on σ_n to solve:

$$f_n = \epsilon \sum_{b=1}^{n-1} \frac{\hat{s}_{nb}}{\sigma_n - \sigma_b} = \frac{\epsilon \sum_{b=1}^{n-1} \hat{s}_{nb} \prod_{k\neq b}^{n-1} (\sigma_n - \sigma_k)}{\prod_{j=1}^{n-1} (\sigma_n - \sigma_j)} = 0 \,. \tag{3.8}$$

Due to the momentum conservation $\sum_{b=1}^{n-1} \hat{s}_{nb} = 0$, the numerator is a degree $n-3$ polynomial in σ_n:

$$\sum_{b=1}^{n-1} \hat{s}_{nb} \prod_{k\neq b}^{n-1} (\sigma_n - \sigma_k) = \sigma_n^{n-2} \left(\sum_{b=1}^{n-1} \hat{s}_{nb} \right) + \sigma_n^{n-3} \left(\sum_{b=1}^{n-1} \sum_{k\neq b} \hat{s}_{nb} \sigma_k \right) + \ldots = \sigma_n^{n-3} (\cdots) + \ldots \,.$$

Therefore, for each solution $\{\sigma_1, \ldots, \sigma_{n-1}\}$, there are $n-3$ solutions of σ_n from Eq. (3.8), such that the total number of solutions to the n-particle SE is:

$$(n-3)(n-4)! = (n-3)! \,. \tag{3.9}$$

We note that this result derived at the soft limit is generally true since the number of solutions to a system of polynomial equations must be an integer, such that it does not change when we continuous move the kinematics away from the soft limit. Of course, when we hit the multilinear boundaries of the kinematics space, the solutions will become degenerate and factorize into two sectors.

Finally, we note that in four dimensions, the $(n-3)!$ solutions fall into $(n-3)$ sectors labeled by $d = 1, 2 \ldots n-3$. The number of solutions in sector d is the Eulerian number $A(n-3, d-1)$. The derivation will be given in Appendix C. More interestingly, only those solutions in the sector $d = k+1$ give nonzero contribution to the N^{k}MHV Yang-Mills and gravity amplitudes. This feature will be further studied in Chapter 6.

3.1.2 Polynomial form

The SE given in the form of Eq. (3.5) has all the variables in the denominator, which is not convenient for further mathematical manipulation. Following Dolan and Goddard [73], we can write the SE into a polynomial form:

$$\tilde{h}_m \equiv \sum_{|S|=m,\, S\subset \mathsf{N}} k_S^2 \sigma_S = 0 \qquad 2 \leqslant m \leqslant n-2 \,, \tag{3.10}$$

where the summation is over order m subsets of $\mathsf{N} = \{1, 2, \ldots, n\}$, and

$$k_S \equiv \sum_{i\in S} k_i \qquad \sigma_S \equiv \prod_{i\in S} \sigma_i \,.$$

Each $\tilde{h}_m$ is a degree m polynomial in σ's. More interestingly, each monomial of $\tilde{h}_m$ has degree m and it is multilinear in σ's.

To derive Eq. (3.10), we first remind that the SE is equivalent to $P^2(z) = 0$, such that it is also equivalent to the vanishing of the following polynomial:

$$F(z) \equiv P^2(z) \prod_{i=1}^{n} (z - \sigma_i) = \sum_{a=1}^{n} \sum_{b=1}^{n} (k_a \cdot k_b) \prod_{i \neq a,b}^{n} (z - \sigma_i) = 0\,. \tag{3.11}$$

From a simple power counting, we find that $F(z)$ is at most degree $n-2$ in z. However, we can show that the coefficients of z^{n-2} and z^{n-3} vanish identically due to the momentum conservation, and $F(z)$ can be transformed into the form:

$$F(z) = \sum_{m=2}^{n-2} (-1)^m z^{n-m-2} \tilde{h}_m\,. \tag{3.12}$$

Thus $F(z) = 0$ imposes $n-3$ equations in Eq. (3.10), which should be equivalent to the SE.

Now we can partially fixed the gauge by choosing $z_1 = \infty$ and $z_n = 0$ such that Eq. (3.10) becomes:

$$h_m \equiv \lim_{z_1 \to \infty,\, z_n \to 0} \frac{\tilde{h}_{m+1}}{z_1} = \sum_{|S|=m,\, S \subset \mathsf{N}'} (k_1 + k_S)^2 \sigma_S = 0 \qquad 1 \leqslant m \leqslant n-3\,, \tag{3.13}$$

where $\mathsf{N}' = \{2, 3, \ldots, n-1\}$. Then Eq. (3.13) is a set of polynomial equations defined on the projective space $\mathbb{C}P^{n-3}$ with homogeneous coordinates $(\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$. Each h_m is a special degree m homogeneous polynomial in that each monomial of h_m is degree m and multilinear in the σ_a's. Following the Bézout theorem, the number of solutions is

$$\prod_{m=1}^{n-3} \deg(h_m) = (n-3)!\,, \tag{3.14}$$

which agrees with the result (3.9) derived from soft limit recursion.

The polynomial form (3.10) is very useful to numerically evaluate the solutions to SE. Besides, it makes manifest several interesting algebraic properties of the SE, for example, $\{\tilde{h}_m\}$ being a representation of the $SL(2, \mathbb{C})$ algebra [73]. More details in direction can be found in [74]. Moreover, Eq. (3.10) forms a basis that possibly can be used to reduce the CHY integrand. Some recent development can be found in [75, 76].

3.2 The integrated CHY formula

Having studied some general properties of the SE, our next job is to localize the integrand $\mathcal{I}_n$ on to the solutions. This can be formally done by a series of delta functions

$\prod \delta(f_a)$. However, since three f_a's are redundant according to Section 3.1.1, we need to delete them from the chain. This has to be done in an elegant way: we need to define a permutation invariant delta function:

$$\prod_{a=1}^{n}{}'\delta(f_a) \tag{3.15}$$

such that this quantity is independent of which three a's we choose to delete. The reason is that the solutions to the SE certainly do not depend on such a choice, so that we do not want anything in our formalism to depend on it.

To derive the form of Eq. (3.15), we start from inspection the two chain of delta functions:

$$\prod_{a\neq i,j,k} \delta(f_a) \qquad \prod_{a\neq\alpha,\beta,\gamma} \delta(f_a) \qquad (i<j<k,\ \alpha<\beta<\gamma)\,,$$

and $\{i,j,k\} \neq \{\alpha,\beta,\gamma\}$ as sets. They actually only differ by a change of variables:

$$\prod_{a\neq i,j,k} \delta(f_a) = \left|\frac{\partial(f_1,f_2,\ldots,\hat{f}_\alpha,\hat{f}_\beta,\hat{f}_\gamma,\ldots,f_n)}{\partial(f_1,f_2,\ldots,\hat{f}_i,\hat{f}_j,\hat{f}_k,\ldots,f_n)}\right| \prod_{a\neq\alpha,\beta,\gamma} \delta(f_a)\,. \tag{3.16}$$

In the Jacobian, we permute $\{f_i, f_j, f_k\}$ in the numerator and $\{f_\alpha, f_\beta, f_\gamma\}$ in the denominator to the front, which leads to:

$$\left|\frac{\partial(f_1,f_2,\ldots,\hat{f}_\alpha,\hat{f}_\beta,\hat{f}_\gamma,\ldots,f_n)}{\partial(f_1,f_2,\ldots,\hat{f}_i,\hat{f}_j,\hat{f}_k,\ldots,f_n)}\right| = -(-1)^{i+j+k+\alpha+\beta+\gamma}\left|\frac{\partial(f_i,f_j,f_k)}{\partial(f_\alpha,f_\beta,f_\gamma)}\right|\,. \tag{3.17}$$

There is a sign dependence since each permutation of two f's leads to a minus sign. This sign dependence holds if the orders of $\{i,j,k\}$ and $\{\alpha,\beta,\gamma\}$ are preserved. Now using the linear relations (3.6) and the implicit function theorem, we get:

$$\begin{aligned}\left|\frac{\partial(f_i,f_j,f_k)}{\partial(f_\alpha,f_\beta,f_\gamma)}\right| &= -\left|\frac{\partial(P,Q,R)}{\partial(f_\alpha,f_\beta,f_\gamma)}\right|\left|\frac{\partial(P,Q,R)}{\partial(f_i,f_j,f_k)}\right|^{-1} \\ &= -\begin{vmatrix}1 & 1 & 1\\ z_\alpha & z_\beta & z_\gamma\\ z_\alpha^2 & z_\beta^2 & z_\gamma^2\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\ z_i & z_j & z_k\\ z_i^2 & z_j^2 & z_k^2\end{vmatrix}^{-1} = -\frac{z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}}{z_{ij}z_{jk}z_{ki}}\,,\end{aligned} \tag{3.18}$$

such that Eq. (3.16) reaches the form

$$(-1)^{i+j+k}z_{ij}z_{jk}z_{ki}\prod_{a\neq i,j,k}\delta(f_a) = (-1)^{\alpha+\beta+\gamma}z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}\prod_{a\neq\alpha,\beta\gamma}\delta(f_a)\,. \tag{3.19}$$

Here we have used the abbreviation:

$$z_{ij} \equiv z_i - z_j\,,$$

which will be used throughout the following chapters. This result legitimates our definition for the permutation invariant delta function:

$$\prod_{a=1}^{n}{}'\delta\left(f_a\right) \equiv (-1)^{i+j+k} z_{ij}z_{jk}z_{ki} \prod_{\substack{a=1 \\ a\neq i,j,k}}^{n} \delta\left(f_a\right) , \tag{3.20}$$

which must be independent of the choice of $\{i, j, k\}$.

Before we can explicitly perform the integral, we have to divide out the volume of the world sheet gauge group $SL(2,\mathbb{C})$. This is taken care by a standard Fadeev-Popov trick. We first pick out three arbitrary indices $\{p, q, r\}$ with $p < q < r$, and rewrite Eq. (3.2) as:

$$\frac{dz_p \wedge dz_q \wedge dz_r}{z_{pq}z_{qr}z_{rp}} \left(\bigwedge_{c\neq p,q,r} dz_c \right) (-1)^{p+q+r} \left[z_{pq}z_{qr}z_{rp} \prod_{a=1}^{n}{}'\delta\left(f_a\right) \mathcal{I}_n \right] . \tag{3.21}$$

We can easily verify that the measure $(dz_p dz_q dz_r)/(z_{pq}z_{qr}z_{rp})$ is $SL(2,\mathcal{C})$ invariant, such that the combination in the bracket should also be invariant under the $SL(2,\mathbb{C})$ transformations on z_p, z_q and z_r. Therefore, we can use this degree of freedom to fix them at three arbitrary points: $\{z_p, z_q, z_r\} = \{\sigma_p, \sigma_q, \sigma_r\}$, namely:

$$\left[z_{pq}z_{qr}z_{rp} \prod_{a=1}^{n}{}'\delta\left(f_a\right) \mathcal{I}_n \right] = \left[\sigma_{pq}\sigma_{qr}\sigma_{rp} \prod_{a=1}^{n}{}'\delta\left(f_a\right) \mathcal{I}_n \right]_{\{z_p,z_q,z_r\}=\{\sigma_p,\sigma_q,\sigma_r\}} . \tag{3.22}$$

Here $\{\sigma_p, \sigma_q, \sigma_r\}$ can be chosen as the three gauge fixed punctures in the solutions of the SE. Subsequently, the integrand is independent of z_p, z_q, and z_r, so that they can be trivially integrated and cancel the volume of $SL(2,\mathbb{C})$:

$$\int \frac{dz_p dz_q dz_r}{z_{pq}z_{qr}z_{rp}} = \mathrm{Vol}\left[SL(2,\mathbb{C})\right] . \tag{3.23}$$

In this sense, we can effectively define the measure as:

$$\frac{dz_1 \ldots dz_n}{\mathrm{Vol}\left[SL(2,\mathbb{C})\right]} = (-1)^{p+q+r} \left(\sigma_{pq}\sigma_{qr}\sigma_{rp}\right) \prod_{\substack{c=1 \\ c\neq p,q,r}}^{n} dz_c . \tag{3.24}$$

Now collecting Eq. (3.20), Eq. (3.22), and Eq. (3.24), we can perform the integration in Eq. (3.1) to explicitly localize the amplitude on the solutions of the SE:

$$\begin{aligned} \boldsymbol{A}_n &= \int \frac{dz_1 \ldots dz_n}{\mathrm{Vol}\left[SL(2,\mathbb{C})\right]} \left(\prod_{a=1}^{n}{}'\delta\left(f_a\right) \mathcal{I}_n \right) \\ &= (-1)^{i+j+k+p+q+r} \sum_{\{\sigma\}\in\mathrm{sol.}} \frac{(\sigma_{ij}\sigma_{jk}\sigma_{ki})(\sigma_{pq}\sigma_{qr}\sigma_{rp})}{\det\left(\Phi^{ijk}_{pqr}\right)} \mathcal{I}_n(\sigma) , \end{aligned} \tag{3.25}$$

where the $n \times n$ symmetric matrix Φ is defined as:

$$\Phi_{ab} = \begin{cases} \dfrac{s_{ab}}{\sigma_{ab}^2} & a \neq b \\ -\displaystyle\sum_{c \neq a} \dfrac{s_{ac}}{\sigma_{ac}^2} & a = b \end{cases} . \tag{3.26}$$

By Φ^{ijk}_{pqr}, we mean the $(n-3) \times (n-3)$ submatrix of Φ with row $\{i, j, k\}$ and column $\{p, q, r\}$ deleted. It is useful to define the *reduced determinant* of Φ as:

$$\det{}'(\Phi) \equiv \frac{(-1)^{i+j+k+p+q+r} \det\left(\Phi^{ijk}_{pqr}\right)}{(\sigma_{ij}\sigma_{jk}\sigma_{ki})(\sigma_{pq}\sigma_{qr}\sigma_{rp})} , \tag{3.27}$$

since this quantity is independent of which three rows and columns are deleted. This property is expected from how we perform the world sheet integration to obtain Eq. (3.25). Nevertheless, we can directly prove it using the SE. This is done in Appendix A, together with proving $\det'(\Phi)$ being permutation invariant. Finally, the integrated form of the CHY formula can be simply written as:

$$\boldsymbol{A}_n = \sum_{\{\sigma\} \in \text{sol.}} \frac{\mathcal{I}_n(\sigma)}{\det'(\Phi)} , \tag{3.28}$$

which is the main result of this subsection.

3.3 The CHY integrands

In the long list of theories that have a CHY formalism, we are goint to study in detail the amplitudes of Yang-Mills, pure gravity and EYM in this work. Now we present the CHY integrands for these theories.

3.3.1 Yang-Mills

According to [34, 35], the integrand for Yang-Mills is:

$$\mathcal{I}_n^{\text{YM}}(k, \epsilon, \sigma) = \left[\frac{\text{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n})}{\sigma_{12}\sigma_{23} \ldots \sigma_{n1}} + (\text{ noncyclic perm. }) \right] \text{Pf}\,'(\Psi) , \tag{3.29}$$

where Ψ is a $2n \times 2n$ matrix composed of four $n \times n$ matrices:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} . \tag{3.30}$$

These matrices have the following forms:

$$A_{ab} = \begin{cases} \dfrac{s_{ab}}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad C_{ab} = \begin{cases} -\dfrac{\sqrt{2}\,\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ \displaystyle\sum_{c \neq a} \dfrac{\sqrt{2}\,\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b \end{cases} . \tag{3.31}$$

Finally, the *reduced Pfaffian* of Ψ is defined as:

$$\mathrm{Pf}'(\Psi) \equiv \frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}\left(\Psi^{ij}_{ij}\right) \qquad 1 \leqslant i < j \leqslant n\,, \tag{3.32}$$

where Ψ^{ij}_{ij} is the submatrix of Ψ with the row $\{i,j\}$ and column $\{i,j\}$ deleted. The reduced Pfaffian is permutation invariant and independent of the choice of $\{i,j\}$, and this is the reason why it stays outside the sum over permutations in Eq. (3.29). The proof is given in Appendix A. However, if we do not delete any rows and columns, we have $\mathrm{Pf}(\Psi) = 0$ since the upper half $(A, -C^T)$ has two null vectors $(1,\ldots,1)$ and $(\sigma_1,\ldots,\sigma_n)$.

In particular, the color ordered Yang-Mills amplitude in arbitrary dimensions has the CHY representation:

$$A_n(12\ldots n) = \sum_{\{\sigma\}\in\text{sol.}} \frac{1}{\det'(\Phi)} \left[\frac{\mathrm{Pf}'(\Psi)}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}}\right] . \tag{3.33}$$

The gauge invariance of this formula is very easy to verify: if we replace one ϵ_a by k_a in $\mathrm{Pf}'(\Psi)$, then there must be two rows and columns identical in Ψ^{ij}_{ij} such that the reduced Pfaffian must vanish. The correctness of this formula has first been proved in [77] using the BCFW recursive relation [44].

3.3.2 Pure gravity

With the color ordered Yang-Mills amplitudes in hand, it is straightforward to write down the CHY formula for the pure gravity: just perform the KLT product. Moreover, since only the Parke-Taylor factors depend on permutations, the summation over permutations in the KLT product only acts them:

$$\begin{aligned} M_n = \sum_{\{\sigma^{(i)}\}} \sum_{\{\sigma^{(j)}\}} & \frac{\mathrm{Pf}'[\Psi(\sigma^{(i)})]\mathrm{Pf}'[\Psi(\sigma^{(j)})]}{\det'[\Phi(\sigma^{(i)})]\det'[\Phi(\sigma^{(j)})]} \\ & \times \left[-\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in S_{n-3}} PT^{(i)}(1,\boldsymbol{\alpha},n,n-1)S[\boldsymbol{\alpha}|\boldsymbol{\beta}]PT^{(j)}(1,\boldsymbol{\beta},n-1,n)\right] , \end{aligned} \tag{3.34}$$

where the two Parke-Taylor factors are:

$$PT^{(i)}(1,\boldsymbol{\alpha},n,n-1) = \left.\frac{1}{\sigma_{1\alpha_2}\sigma_{\alpha_2\alpha_3}\ldots\sigma_{\alpha_{n-2}n}\sigma_{n,n-1}\sigma_{n-1,1}}\right|_{\sigma=\sigma^{(i)}}$$
$$PT^{(j)}(1,\boldsymbol{\beta},n-1,n) = \left.\frac{1}{\sigma_{1\beta_2}\sigma_{\beta_2\beta_3}\ldots\sigma_{\beta_{n-2},n-1}\sigma_{n-1,n}\sigma_{n1}}\right|_{\sigma=\sigma^{(j)}}.$$

However, Eq. (3.34) does not seems to be compatible with our scheme (3.28) since it involves a double summation over the solutions. In a seminal paper that established the beauty of this formalism [33], CHY proved the *KLT orthogonality* of the solutions:

$$-\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in S_{n-3}} PT^{(i)}(1,\boldsymbol{\alpha},n,n-1)S[\boldsymbol{\alpha}|\boldsymbol{\beta}]PT^{(j)}(1,\boldsymbol{\beta},n-1,n) = \det{}'[\Phi(\sigma^{(i)})]\delta^{ij}\,, \tag{3.35}$$

namely, if we view each Parke-Taylor factor as a vector in the $(n-3)!$ dimensional solution space, they are orthogonal with respect to the KLT kernel. As a result, the CHY formula for pure gravity significantly simplifies to:

$$M_n = \sum_{\{\sigma\}\in\text{sol.}} \frac{\text{Pf}'(\Psi)\times\text{Pf}'(\Psi)}{\det{}'(\Phi)}\,. \tag{3.36}$$

This resembles the double copy relation (2.105): we just need to trade one "color factor"

$$\frac{\text{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}}$$

by another "kinematic numerator" $\text{Pf}'(\Psi)$ to obtain a gravity amplitude. Recent studies on the C-K duality in the context of CHY can be found in [35, 78, 79].

3.3.3 Einstein-Yang-Mills

The CHY formalism for general multitrace EYM amplitudes can be found in [37]. In this work, we only focus on the n-point single trace EYM amplitudes with r gluons and $s=n-r$ gravitons. This prescription is first given in [36]. In our notation, we use H and G to denote the set of gravitons and gluons:

$$\mathsf{G}\equiv\{1,2,\ldots,r\}\qquad \mathsf{H}\equiv\{r+1,r+2,\ldots,r+s\}\,. \tag{3.37}$$

The color ordered single trace EYM amplitude can then be written as:

$$M_{n,r}(12\ldots r|\mathsf{H}) = \sum_{\{\sigma\}\in\text{sol.}}\frac{1}{\det{}'(\Phi)}\frac{\text{Pf}\,(\Psi_{\mathsf{H}})\,\text{Pf}'(\Psi)}{\sigma_{12}\sigma_{23}\ldots\sigma_{r1}}\qquad(2\leqslant r\leqslant n)\,. \tag{3.38}$$

Each gluon carries a polarization vector ϵ_i^μ, while each graviton carries a polarization tensor $\epsilon_i^\mu \widetilde{\epsilon}_i^\nu$. Thus we have two sets of polarizations involved in EYM amplitudes:

$$\{\epsilon\} \equiv \{\epsilon_1, \ldots, \epsilon_n\} \qquad \{\widetilde{\epsilon}\} \equiv \{\widetilde{\epsilon}_{r+1} \ldots, \widetilde{\epsilon}_{r+s}\}\,.$$

According to Eq. (3.38), the n polarizations in the set $\{\epsilon\}$ are encoded in the matrix Ψ, defined as in Eq. (3.30), while the other s polarizations in $\{\widetilde{\epsilon}\}$ are contained in the $2s \times 2s$ matrix Ψ_{H}, defined as:

$$\Psi_{\mathsf{H}} = \begin{pmatrix} A_{\mathsf{H}} & -C_{\mathsf{H}}^T \\ C_{\mathsf{H}} & B_{\mathsf{H}} \end{pmatrix}, \tag{3.39}$$

where each component is just the submatrix of the corresponding one in Ψ whose indices are in the graviton set H:

$$A_{\mathsf{H}} \equiv A^{12\ldots r}_{12\ldots r} \qquad B_{\mathsf{H}} \equiv B^{12\ldots r}_{12\ldots r} \qquad C_{\mathsf{H}} \equiv C^{12\ldots r}_{12\ldots r}\,. \tag{3.40}$$

Gluon polarizations do not appear in Ψ_{H}, while gluon momenta appear only in the diagonal elements of C_{H}:

$$(C_{\mathsf{H}})_{aa} = \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{\sqrt{2}\,\widetilde{\epsilon}_a \cdot k_b}{\sigma_{ab}}\,.$$

The Bose symmetry in the gravitons is realized trivially in Eq. (3.38). Exchanging two gravitons only leads to a switch of two pairs of rows and columns in Ψ and Ψ_{H}, which apparently leaves the amplitude invariant.

If we have $n-1$ gravitons, Eq. (3.38) is not well defined because no proper Parke-Taylor factor exists for a single gluon. This amplitude should vanish since emitting a single gluon is forbidden by the color conservation. Thus, we can define that the CHY integrand vanishes for $n-1$ gravitons. Finally, if all the n particles are gravitons, we just return to Eq. (3.36). There is no consistent way to reduce the number of gluons to zero in EYM. This can be understood from the fact that heterotic and closed string have different world sheet topology.

CHAPTER 4

SPECIAL RATIONAL SOLUTION AND MHV AMPLITUDES

In principle, the solutions to the SE are very complicated algebraic functions of the Mandelstam variables. However, in four dimensions, two special solutions exist that are rational in spinor variables, and related through a complex conjugation:

$$\sigma_a^{(1)} = \frac{\langle a\eta\rangle\langle\theta\xi\rangle}{\langle a\xi\rangle\langle\theta\eta\rangle}, \qquad \sigma_{ab}^{(1)} = \frac{\langle ab\rangle\langle\theta\xi\rangle\langle\eta\xi\rangle}{\langle a\xi\rangle\langle b\xi\rangle\langle\theta\eta\rangle}, \tag{4.1}$$

$$\sigma_a^{(2)} = \frac{[a\eta][\theta\xi]}{[a\xi][\theta\eta]}, \qquad \sigma_{ab}^{(2)} = \frac{[ab][\theta\xi][\eta\xi]}{[a\xi][b\xi][\theta\eta]}, \tag{4.2}$$

where the projective spinor η, θ and ξ encode the $SL(2,\mathbb{C})$ redundancy in this system, as discussed in Section 3.1.1. For example, if we choose $\eta = n-2$, $\theta = n-1$ and $\xi = n$, we can put:

$$\sigma_{n-2}^{(1)} = \sigma_{n-2}^{(2)} = 0 \qquad \sigma_{n-1}^{(1)} = \sigma_{n-1}^{(2)} = 1 \qquad \sigma_n^{(2)} = \sigma_n^{(2)} = \infty\,.$$

Written in terms of momentum components, this solution was already known in the first appearance of the SE [65, 66]. However, this spinor form was first given in [80]. Since Eq. (4.1) and Eq. (4.2) depend only on angular and square brackets, respectively, people immediately conjecture that they should correspond to the MHV and anti-MHV amplitudes, for example, see [63, 81]. In our work [39], we proved this connection analytically: Eq. (4.1) supports the MHV amplitudes while Eq. (4.2) supports the anti-MHV amplitudes. We will see in this chapter that once plugged in by the solution (4.1), both $\mathrm{Pf}'(\Psi)$ and $\det'(\Phi)$ will take a very nice form: the world sheet $SL(2,\mathbb{C})$ dependency factorizes out of the gauge invariant building blocks of physical amplitudes completely. Then using the prescription Eq. (3.29) and Eq. (3.36), we can explicitly derive the Parke-Taylor formula for Yang-Mills MHV and Hodges formula [50] for gravity MHV amplitudes. This chapter presents the derivation, which gives essential experience and insight to work with other solutions and amplitudes. In this hindsight, we call Eq. (4.1) *the MHV solution* and Eq. (4.2) *the anti-MHV solution*. In Section 4.1 and Section 4.2, we derive the expressions for $\det'(\Phi)$ and $\mathrm{Pf}'(\Psi)$ evaluated on the MHV solution. Then in Section 4.3, we present the resultant MHV amplitudes. Finally, we discuss and remark on the contribution of other solutions in Section 4.4.

4.1 Calculation of the reduced determinant

If we plug in the MHV solution to matrix Φ given in Eq. (3.26), we have:

$$\begin{aligned} \Phi_{ab} &= -\frac{[ab]\langle a\xi\rangle^2\langle b\xi\rangle^2\langle\theta\eta\rangle^2}{\langle ab\rangle\langle\eta\xi\rangle^2\langle\theta\xi\rangle^2} && (a \neq b) \\ \Phi_{aa} &= \frac{\langle a\xi\rangle^2\langle\theta\eta\rangle^2}{\langle\eta\xi\rangle^2\langle\theta\xi\rangle^2}\sum_{l\neq a}\frac{[al]\langle l\xi\rangle^2}{\langle al\rangle} && (\text{diagonal elements})\,. \end{aligned} \tag{4.3}$$

In the off-diagonal elements, there are a common factor $\langle a\xi\rangle^2$ and $\langle b\xi\rangle^2$ to each row and column, such that it is tempting to pull them out of the determinant. To do this, we need a factor $\langle a\xi\rangle^4$ in the diagonal elements, so that we need to rewrite the summation into a better form to make the factor $\langle a\xi\rangle^4$ manifest:

$$\begin{aligned} \sum_{l\neq a}\frac{[al]\langle l\xi\rangle^2}{\langle al\rangle} = \sum_{l\neq a}\frac{[al]\langle l\xi\rangle^2\langle a\eta\rangle}{\langle al\rangle\langle a\eta\rangle} &= \sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle\eta\xi\rangle}{\langle a\eta\rangle} + \langle a\xi\rangle^2\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\eta\rangle} \\ &= \langle a\xi\rangle^2\sum_{l\neq a}\frac{[al]\langle l\xi\rangle\langle l\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\eta\rangle}\,, \end{aligned}$$

where the first summation over l vanishes due to the momentum conservation. Therefore, the matrix elements of Φ can be related to those of the Hodges matrix ϕ defined in Eq. (2.94) as:

$$\Phi_{ab} = -\frac{\langle a\xi\rangle^2\langle b\xi\rangle^2\langle\theta\eta\rangle^2}{\langle\eta\xi\rangle^2\langle\theta\xi\rangle^2}\phi_{ab} \quad (a\neq b)\,, \qquad \Phi_{ab} = \frac{\langle a\xi\rangle^4\langle\theta\eta\rangle^2}{\langle\eta\xi\rangle^2\langle\theta\xi\rangle^2}\phi_{aa}\,, \tag{4.4}$$

such that all the prefactors can be pulled out when calculating the determinant. Consequently, we have:

$$\det(\Phi^{ijk}_{pqr}) = (-1)^{n-3}(F_{\eta\theta\xi})^{2n-6}(P_\xi)^4(d_{ijk}d^{pqr})^2\det(\phi^{ijk}_{pqr})\,, \tag{4.5}$$

where the F, P and d symbols are defined as

$$F_{\eta\theta\xi} = \frac{\langle\theta\eta\rangle}{\langle\eta\xi\rangle\langle\theta\xi\rangle} \qquad P_\xi = \prod_{a=1}^{n}\langle a\xi\rangle \qquad d_{abc} = d^{abc} = \frac{1}{\langle a\xi\rangle\langle b\xi\rangle\langle c\xi\rangle}\,. \tag{4.6}$$

As another ingredient of the reduced determinant, we have:

$$\sigma_{ij}\sigma_{jk}\sigma_{ki}\sigma_{pq}\sigma_{qr}\sigma_{rp} = (F_{\eta\theta\xi})^{-6}(c_{ijk}c^{pqr})^{-1}(d_{ijk}d^{pqr})^2\,, \tag{4.7}$$

where c symbols have been defined in Eq. (2.93). Combining Eq. (4.5) and Eq. (4.7), we get the final result of the reduced determinant $\det'(\Phi)$ evaluated on the MHV solution (4.1):

$$\det'(\Phi) = \frac{(-1)^{i+j+k+p+q+r}\det\left(\Phi^{ijk}_{pqr}\right)}{(\sigma_{ij}\sigma_{jk}\sigma_{ki})(\sigma_{pq}\sigma_{qr}\sigma_{rp})} = (F_{\eta\theta\xi})^{2n}(P_\xi)^4\bar{M}(12\ldots n)\,, \tag{4.8}$$

where the reduced gravity amplitude $\bar{M}$ has been defined in Eq. (2.92). In this equation, we can clearly see that the world sheet $SL(2,\mathbb{C})$ dependent parts factorize out of the physical gauge invariant quantity $\bar{M}$. Since we have proved in Appendix A that $\det'(\Phi)$ is independent of the deleted rows and columns, so does the reduced gravity amplitude $\bar{M}$. For other solutions, $\det'(\Phi)$ can be viewed as the generalization of $\bar{M}$ to other helicity configurations.

4.2 Calculation of the reduced Pfaffian

Before we start to calculate $\mathrm{Pf}'(\Psi)$, we need to fix the gauge in the polarization vectors such that the structure of matrix Ψ is made as simple as possible. For the MHV configuration $(1^-2^-3^+\ldots n^+)$, the best gauge choice is:

$$(\epsilon_a^-)_\mu = \frac{\langle a|\overline{\sigma}_\mu|n]}{\sqrt{2}\,[na]}\quad (a=1,2)\,,\qquad (\epsilon_a^+)_\mu = \frac{[a|\sigma_\mu|1\rangle}{\sqrt{2}\,\langle 1a\rangle}\quad (a=3,4\ldots n)\,, \tag{4.9}$$

such that the only nonzero inner products between polarization vectors are $\epsilon_2\cdot\epsilon_a^+$ (except for $a=n$). In other words, all nonzero elements in the matrix B are in the second row and column. Moreover, this gauge choice also leads to $k_1\cdot\epsilon_a^+=0$ and $k_n\cdot\epsilon_a^-=0$, which leads to additional zero elements in the matrix C. Therefore, before any manipulation, the shape of matrix Ψ is shown in Figure 4.1. For convenience, we deleted the $(n-1)$-th and n-th row and column of Ψ when calculating the reduced Pfaffian:

$$\mathrm{Pf}'(\Psi) = \frac{-1}{\sigma_{n-1,n}}\mathrm{Pf}\left(\Psi^{n-1,n}_{n-1,n}\right) \equiv \frac{-1}{\sigma_{n-1,n}}\mathrm{Pf}\,(\widetilde{\Psi})\,. \tag{4.10}$$

As proved in Appendix A, the value of $\mathrm{Pf}'(\Psi)$ does not depend on such a choice.

To evaluate $\mathrm{Pf}\,(\widetilde{\Psi})$, we can expand it along one row using the formula:

$$\mathrm{Pf}\,(X) = \sum_{\substack{j=1\\ j\neq i}}^{2N}(-1)^{i+j+1+\theta(i-j)}x_{ij}\mathrm{Pf}\left(X^{ij}_{ij}\right)\,,\qquad \theta(i-j) = \left\{\begin{array}{ll} 1 & i>j \\ 0 & i<j\end{array}\right.\,, \tag{4.11}$$

for a $2N\times 2N$ antisymmetric matrix $X=(x_{ij})$. After this expansion, we would like the remaining submatrix to contain as many zeros as possible, so that further simplification

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} =$$

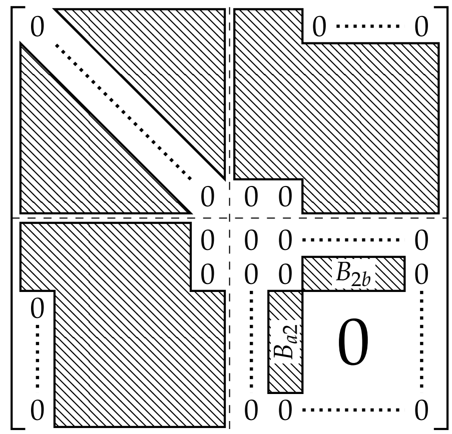


Figure 4.1. The shape of matrix Ψ. Here we show the structure of Ψ after we fixed the gauge (4.9). Only the shaded regions are nonzero.

can be straightforward. Following this guideline, we expand $\widetilde{\Psi}$ along its n-th row, which is the 2nd row of the matrix B and C:

$$\mathrm{Pf}\left(\widetilde{\Psi}\right) = \sum_{\substack{b=1 \\ b\neq n}}^{2n-2} (-1)^{n+b+1+\theta(n-b)} \widetilde{\Psi}_{nb} \mathrm{Pf}\left(\widetilde{\Psi}^{nb}_{nb}\right). \tag{4.12}$$

Depending on the value of b, we have:

$$\widetilde{\Psi}_{nb} = \begin{cases} C_{2b} & 1 \leqslant b \leqslant n-2 \\ B_{2,b-n+2} & n-1 \leqslant b \leqslant 2n-2 \end{cases}. \tag{4.13}$$

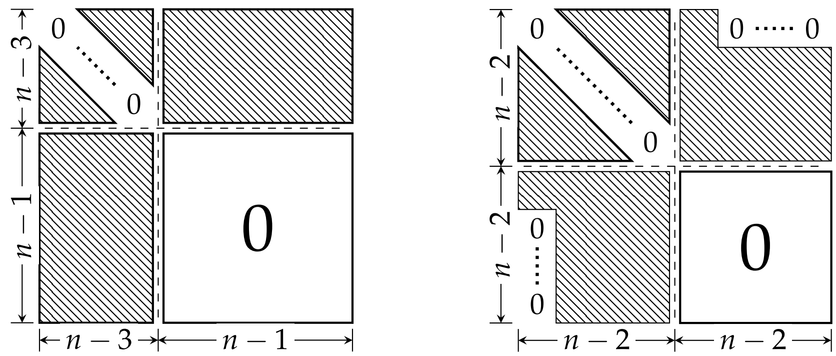
Clearly, when $1 \leqslant b \leqslant n-2$, the b-th row and column deleted are in the C part, such that the submatrix $\widetilde{\Psi}_{nb}$ has the shape displayed in Figure 4.2a. In this case, the lower left C part has dimension $(n-1) \times (n-3)$. As a result, we can always find an elementary transformation that makes at least two rows in this region zero, such that we have at least two rows of zero in $\widetilde{\Psi}_{nb}$. Therefore, we have $\mathrm{Pf}\left(\widetilde{\Psi}^{nb}_{nb}\right) = 0$ in this region, and we can rewrite the expansion (4.12) as:

$$\mathrm{Pf}\left(\widetilde{\Psi}\right) = \sum_{m=3}^{n} (-1)^{m+1} B_{2m} \mathrm{Pf}\left(\widetilde{\Psi}^{n,m+n-2}_{n,m+n-2}\right) \equiv \sum_{m=3}^{n} (-1)^{m+1} B_{2m} \mathrm{Pf}\left(\psi_m\right), \tag{4.14}$$

where the shape of ψ_m is shown in Figure 4.2b.

Next, we apply the expansion (4.11) to each ψ_m along the $(n-1)$-th row, which is the 1-st row of the matrix C:

$$\mathrm{Pf}\left(\psi_m\right) = \sum_{s=1}^{n-2} (-1)^{n+s+1} C_{1s} \mathrm{Pf}\left([\psi_m]^{n-1,s}_{n-1,s}\right). \tag{4.15}$$



(a) $1 \leqslant b \leqslant n-2$

(b) $n-1 \leqslant b \leqslant 2n-2$

Figure 4.2. The shape of matrix $\widetilde{\Psi}^{nb}_{nb}$. In this figure, we display the structure of $\widetilde{\Psi}^{nb}_{nb}$: (a) The submatrix of $\widetilde{\Psi}$ when the deleted column is in the C part during the expansion of $\mathrm{Pf}\,(\widetilde{\Psi})$. The Pfaffian of this submatrix is zero. (b) The submatrix of $\widetilde{\Psi}$ obtained when the deleted column is in the B part during the expansion of $\mathrm{Pf}\,(\widetilde{\Psi})$.

In this expansion, we can similarly show that all the submatrix $[\psi_m]^{n-1,s}_{n-1,s}$ with $s \geqslant 2$ must have zero Pfaffian. We can perform the elementary transformation shown in Figure 4.3 for those submatrices, which only changes their Pfaffians by a sign. After that, we see an $(n-2)\times(n-2)$ block of zeros at the right bottom corner, while the off-diagonal block at the left bottom is $(n-2)\times(n-4)$ dimensional. Therefore, we can always find another elementary transformation in the lower part of the matrix that makes at least two entire rows zero, which consequently leads to a zero Pfaffian. Then the summation in Eq. (4.15) only contains one term:

$$\mathrm{Pf}\,(\psi_m) = (-1)^n C_{11} \mathrm{Pf}\left([\psi_m]^{n-1,1}_{n-1,1}\right) \equiv (-1)^n C_{11} \mathrm{Pf}\,(\psi'_m)\,, \tag{4.16}$$

where ψ'_m is an $(2n-6)\times(2n-6)$ matrix. Now it is a good time to see how ψ'_m is made from the original A, B and C matrix:

$$\psi'_m = \left(\begin{array}{c:c} A^{1,n-1,n}_{1,n-1,n} & -\left(C^{1,2,m}_{1,n-1,n}\right)^T \\ \hdashline C^{1,2,m}_{1,n-1,n} & 0 \end{array}\right). \tag{4.17}$$

If the lower left block $C^{1,2,m}_{1,n-1,n}$ does not have full rank, we can find an elementary transformation to make one entire row zero in this part such that $\mathrm{Pf}\,(\psi'_m) = 0$. On the other hand,

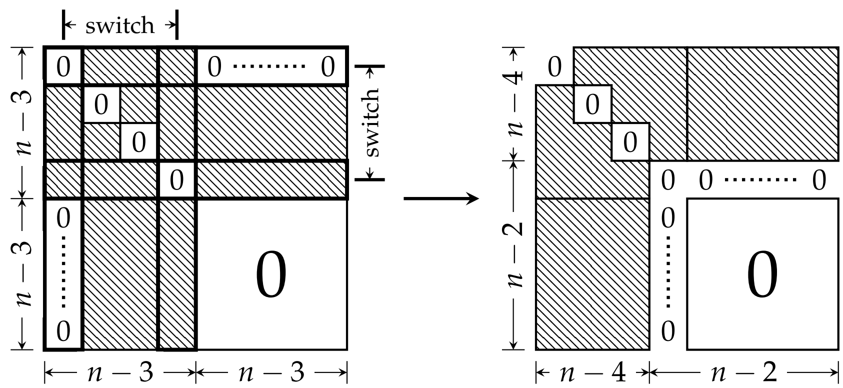


Figure 4.3. The shape of $[\psi_m]^{n-1,s}_{n-1,s}$ with $s \geqslant 2$. The $(2n-6)\times(2n-6)$ submatrix of ψ_m when the deleted column is not the first one during the expansion. To tell the Pfaffian of this submatrix is zero, one can switch the two rows and columns as indicated.

if $C^{1,2,m}_{1,n-1,n}$ has full rank, we can find another elementary transformation to make the entire $A^{1,n-1,n}_{1,n-1,n}$ matrix zero. In both cases, we can write:

$$\mathrm{Pf}\left(\psi'_m\right) = (-1)^{\frac{(n-2)(n-3)}{2}} \det\left(C^{12m}_{1,n-1,n}\right). \tag{4.18}$$

Now collecting Eq. (4.10), Eq. (4.14), Eq. (4.16), and Eq. (4.18), we can express the expansion of $\mathrm{Pf}'(\Psi)$ as:

$$\mathrm{Pf}'(\Psi) = (-1)^{\frac{(n-2)(n-3)}{2}} \sum_{m=3}^{n} (-1)^{m+n+1} B_{2m} C_{11} \left[\frac{-1}{\sigma_{n-1,n}} \det\left(C^{12m}_{1,n-1,n}\right)\right]. \tag{4.19}$$

If we divide the matrix C into two parts $C_\pm$ according to the helicities, we find that $C^{1,2,m}_{1,n-1,n}$ is actually a submatrix of C_+. Then Eq. (4.19) implies that to make $\mathrm{Pf}'(\Psi) \neq 0$ at MHV, we must have $\mathrm{rank}(C_+) = n-3$. This is the first hint that in four dimensions, the solutions to SE and the rank of the matrix C may have an interesting relation. We will discuss this point in detail in Chapter 6.

The next task is thus to plug in the MHV solution (4.1) into Eq. (4.19). First, the relevant C matrix elements are:

$$\begin{aligned} C_{ab} &= \frac{[ab]\langle a\xi\rangle\langle b\xi\rangle\langle b1\rangle\langle\theta\eta\rangle}{\langle ab\rangle\langle a1\rangle\langle\theta\xi\rangle\langle\eta\xi\rangle} && (3 \leqslant a \leqslant n \text{ and } b \neq a), \\ C_{aa} &= -\frac{\langle a\xi\rangle^2\langle\theta\eta\rangle}{\langle\theta\xi\rangle\langle\eta\xi\rangle} \sum_{\substack{l=1 \\ l\neq a}}^{n} \frac{[al]\langle l\xi\rangle\langle l1\rangle}{\langle al\rangle\langle a\xi\rangle\langle a1\rangle} && (3 \leqslant a \leqslant n). \end{aligned} \tag{4.20}$$

Plugging them into the determinant of $C^{1,2,m}_{1,n-1,n}$, we get:

$$\det\left(C^{12m}_{1,n-1,n}\right) = (F_{\eta\theta\xi})^{n-3}(P_\xi)^2 d_{12m} d^{1,n-1,n} \frac{\langle 12\rangle\langle 1m\rangle}{\langle 1,n-1\rangle\langle 1n\rangle} \det\left(\phi^{12m}_{1,n-1,n}\right).$$

The origin of those coefficients in front of the Hodges minor $\det(\phi^{1,2,m}_{1,n-1,n})$ is not difficult to identify: first, $F_{\eta\theta\xi}$ is a factor common to all the $n-3$ rows, so that it is raised to this power; $(P_\xi)^2 d_{12m} d^{1,n-1,n}$ comes from pulling out $\langle a\xi\rangle$ and $\langle b\xi\rangle$ that are common to each row and column respectively, where the d's are to compensate the deleted rows and columns; finally, when pulling out $1/\langle a1\rangle$ and $\langle b1\rangle$ from each row and column, we need the factor $\langle 12\rangle\langle 1m\rangle/(\langle 1,n-1\rangle\langle 1n\rangle)$ to compensate the deleted rows and columns. Now grouping the above result with $-1/\sigma_{n-1,n}$, we can trade one d factor into one of the c's that is required for a reduced gravity amplitude $\bar{M}$:

$$\frac{-\det(C^{12m}_{1,n-1,n})}{\sigma_{n-1,n}} = (F_{\eta\theta\xi})^{n-2}(P_\xi)^2 d_{12m} c^{1,n-1,n} \det\left(\phi^{12m}_{1,n-1,n}\right)\frac{\langle 12\rangle\langle 1m\rangle}{\langle 1\xi\rangle}\,.$$

The other d is actually transformed into c with the help of B_{2m}:

$$(-1)^{m+n+1}B_{2m} = (-1)^{m+n+1}\frac{\epsilon_2\cdot\epsilon_m}{\sigma_{2m}} = F_{\theta\eta\xi}\langle 12\rangle^2(-1)^{m+n+1}\frac{c_{12m}}{d_{12m}}\frac{[mn]}{\langle 1\xi\rangle[n2]}\,. \tag{4.21}$$

Interestingly, the alternating sign $(-1)^m$ gets grouped into the m-independent $\bar{M}$ such that the summation over m resutls in:

$$\begin{aligned}
&\sum_{m=3}^{n}(-1)^{m+n+1}B_{2m}\left[\frac{-1}{\sigma_{n-1,n}}\det\left(C^{12m}_{1,n-1,n}\right)\right] \\
&= -(F_{\theta\eta\xi})^{n-1}(P_\xi)^2\frac{\langle 12\rangle^3}{\langle 1\xi\rangle^2}\bar{M}(12\ldots n)\sum_{m=3}^{n}\frac{\langle 1m\rangle[mn]}{[n2]} \\
&= -(F_{\theta\eta\xi})^{n-1}(P_\xi)^2\frac{\langle 12\rangle^4}{\langle 1\xi\rangle^2}\bar{M}(12\ldots n)\,.
\end{aligned} \tag{4.22}$$

Finally, combining with C_{11}:

$$C_{11} = \sum_{b=2}^{n}\frac{\sqrt{2}\,\epsilon_1\cdot k_b}{\sigma_{1b}} = (F_{\theta\eta\xi})\langle 1\xi\rangle^2\,,$$

we arrive at the final result of $\mathrm{Pf}'(\Psi)$ evaluated on the MHV solution (4.1):

$$\mathrm{Pf}'(\Psi) = -(-1)^{\frac{(n-2)(n-3)}{2}}(F_{\theta\eta\xi})^n(P_\xi)^2\langle 12\rangle^4\bar{M}(12\ldots n)\,. \tag{4.23}$$

Similar to Eq. (4.8), The final result features a complete separation of the world sheet $SL(2,\mathbb{C})$ dependent factor and the physical gauge invariant component. In particular, $\mathrm{Pf}'(\Psi)$ reproduces the correct Parke-Taylor numerator $\langle 12\rangle^4$. If we had started with a less convenient gauge, we would reach the same result. The gauge freedom is encoded in ϕ_{aa} contained in $\bar{M}$, and we have already proved it in Eq. (2.95).

Finally, we note that if the two negative helicity gluons are at the position i and j, Eq. (4.19) has a generalized expression [39]:

$$\mathrm{Pf}'(\Psi) = -(-1)^{\frac{(n-2)(n-3)}{2}} \sum_{\substack{m=1\\ m\neq i,j}}^{n} \mathrm{perm}(ijm)\mathrm{perm}(ipq) B_{jm} C_{ii} \left[\frac{1}{\sigma_{pq}} \det\left(C^{ijm}_{ipq}\right)\right], \tag{4.24}$$

where $\mathrm{perm}(ijm)$ is the permutation signature of $\{i, j, m, 1, 2 \dots\}$ with respect to the identity $\{1, 2 \dots n\}$. After plugging in the MHV solution (4.1), the result is:

$$\mathrm{Pf}'(\Psi) = (-1)^{\frac{(n-1)(n-4)}{2}} (F_{\theta\eta\xi})^n (P_\xi)^2 \langle ij\rangle^4 \bar{M}(12\dots n). \tag{4.25}$$

Comparing with Eq. (4.23), we only need to replace the Parke-Taylor numerator $\langle 12\rangle^4$ by the general one $\langle ij\rangle^4$. We have also absorbed a minus sign into the power of (-1).

4.3 Yang-Mills and gravity MHV from CHY

The final piece in the Yang-Mills integrand is the Parke-Taylor factor. When evaluated on the MHV solution (4.1), it simply equals:

$$PT(12\dots n) = \frac{1}{\sigma_{12}\sigma_{23}\dots\sigma_{n1}} = \frac{(F_{\theta\eta\xi})^n (P_\xi)^2}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}. \tag{4.26}$$

Noticeably, the world sheet $SL(2,\mathbb{C})$ dependence of PT is exactly the same as that of $\mathrm{Pf}'(\Psi)$, see Eq. (4.23). Then using the CHY formula for color ordered Yang-Mills amplitude (3.33) with Eq. (4.8) Eq. (4.23), and Eq. (4.26), we can reproduce the correct Parke-Taylor formula for MHV, up to an inconsequential overall factor:

$$A_n(1^-2^-3^+\dots n^+) = \frac{PT(12\dots n)\mathrm{Pf}'(\Psi)}{\det'(\Phi)} = (-1)^{\frac{(n-1)(n-4)}{2}} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}. \tag{4.27}$$

In this equation, the world sheet $SL(2,\mathbb{C})$ dependent factors $F_{\theta\eta\xi}$ and P_ξ in the three building blocks nicely cancel each other. Moreover, the reduced gravity amplitude $\bar{M}$, contained in both $\mathrm{Pf}'(\Psi)$ and $\det'(\Phi)$, also gets canceled.

Since PT and $\mathrm{Pf}'(\Psi)$ have the same world sheet $SL(2,\mathbb{C})$ dependence, replacing the former by the latter does not spoil the cancellation of such a dependence. According to the CHY formulism, this operation results in the gravity MHV amplitude:

$$M_n(1^-2^-3^+\dots n^+) = \frac{\mathrm{Pf}'(\Psi)\times\mathrm{Pf}'(\Psi)}{\det'(\Phi)} = \langle 12\rangle^8 \bar{M}(12\dots n). \tag{4.28}$$

Indeed, we obtain correctly the Hodges formula.

We note that by exchanging all the angular and square brackets, we can obtain the anti-MHV Yang-Mills and gravity amplitudes along the same line of derivation. In other words, the anti-MHV solution (4.2) can reproduce the correct anti-MHV Yang-Mills and gravity amplitudes in the same way.

4.4 Remarks and summary

As we have argued in Section 4.2, we have $\mathrm{Pf}'(\Psi) \neq 0$ at MHV configurations if and only if $\mathrm{rank}(C_+) = n-3$. The MHV solution explicitly does so by leading to the correct Parke-Taylor formula and Hodges formula for Yang-Mills and gravity MHV amplitudes. It is thus tempting to check what happens if we use the anti-MHV solution (4.2) into Eq. (4.19). In this case, the C_+ part elements are:

$$
\begin{aligned}
C_{ab} &= \frac{[a\xi][b\xi]\langle b1\rangle[\theta\eta]}{\langle a1\rangle[\theta\xi][\eta\xi]} && (3 \leqslant a \leqslant n \text{ and } b \neq a) \\
C_{aa} &= \frac{[a\xi]^2[\theta\eta]}{[\theta\xi][\eta\xi]} && (3 \leqslant a \leqslant n)\,.
\end{aligned}
\tag{4.29}
$$

It is now easy to verify that all the rows are proportional, since

$$
\frac{\langle a1\rangle}{[a\xi]} \times C_{ab} \;\text{ is independent of } a.
$$

Therefore, the anti-MHV solution leads to $\mathrm{rank}(C_+) = 1$, and it does not contribute to MHV amplitudes. Similarly, all the other solutions make $\mathrm{rank}(C_+) < n-3$ such that only the MHV solution (4.1) contributes to MHV amplitudes, as the name indicates.

To summarize, in this chapter, we have proved analytically that the MHV solution (4.1) does lead to the correct Parke-Taylor formula for Yang-Mills and Hodges for gravity at MHV, as in Eq. (4.27) and Eq. (4.28). We claim that all the other solutions do not contribute to MHV amplitudes since they make $\mathrm{rank}(C_+) < n-3$. This point will be further clarified later in Chapter 6.

CHAPTER 5

SINGLE TRACE EINSTEIN-YANG-MILLS MHV AMPLITUDES

In the previous chapter, we derived the Parke-Taylor and Hodges formula for Yang-Mills and gravity MHV amplitudes. We expect that such techniques may also lead to simple and compact expression for MHV amplitudes of other theories. Einstein-Yang-Mills (EYM) is such an immediate generalization. Indeed, we show in [40] that the CHY formalism can give a much simpler expression for this amplitude, compared to what exists in the literature. To be specific, we are looking at the single trace (gluon) color ordered amplitude $M_{n,r}(12\ldots r|\mathsf{H})$, in which gluon and graviton set are:

$$\mathsf{G}=\{1,2\ldots r\}\,,\qquad \mathsf{H}=\{r+1,r+2\ldots r+s\}\,,\quad (n=r+s)\,. \tag{5.1}$$

As before, we use $\mathsf{N}=\{1,2\ldots n\}=\mathsf{G}\cup\mathsf{H}$ to denote the set of all external particles. In addition, it is convenient to define the sets of positive and negative helicity gravitons $\mathsf{H}_\pm$ and similarly gluons $\mathsf{G}_\pm$, whose orders are $|\mathsf{H}_\pm|=s^\pm$ and $|\mathsf{G}_\pm|=r^\pm$.

The CHY integrand in this case is given by Eq. (3.38), repeated here as:

$$M_{n,r}(12\ldots r|\mathsf{H})=\sum_{\{\sigma\}\in\text{sol.}}\frac{1}{\det'(\Phi)}\frac{\text{Pf}\,(\Psi_\mathsf{H})\,\text{Pf}\,'(\Psi)}{\sigma_{12}\sigma_{23}\ldots\sigma_{r1}}\qquad(2\leqslant r\leqslant n)\,, \tag{5.2}$$

where $\text{Pf}\,(\Psi_\mathsf{H})$ is defined in Eq. (3.39). At MHV, since the integrand contains $\text{Pf}\,'(\Psi)$, the support can only be the MHV solution (4.1). As calculated in the previous chapter, we already have:

$$\frac{\text{Pf}\,'(\Psi)}{\det'(\Phi)}=\frac{(-1)^{\frac{(n-1)(n-4)}{2}}\langle ij\rangle^4}{(F_{\theta\eta\xi})^n(P_\xi)^2}\qquad\qquad \frac{1}{\sigma_{12}\ldots\sigma_{r1}}=\frac{(F_{\theta\eta\xi})^r}{\langle 12\rangle\ldots\langle r1\rangle}\prod_{a=1}^{r}\langle a\xi\rangle^2\,, \tag{5.3}$$

if the two negative helicity particles are i and j, be it gluons or gravitons.[1] Thus, the main subject of this chapter is to calculate $\text{Pf}\,(\Psi_\mathsf{H})$. The outcome will be different depending on the nature of negative helicity particles, and we have the following three cases:

[1]Namely, we have $1\leqslant i<j\leqslant n$.

1. Two negative helicity gluons, hereafter, (g^-g^-) amplitude.
2. One negative helicity gluon and the other graviton, hereafter, (g^-h^-) amplitude.
3. Two negative helicity gravitons, hereafter, (h^-h^-) amplitude.

We first derive the expressions for (g^-g^-) and (g^-h^-) MHV amplitudes in Section 5.1. Then in Section 5.2, we prove that the (h^-h^-) amplitude must identically vanish. Finally, we prove that our new results, which are much simpler, agree with the existing ones in the literature, using a graph theoretical approach discussed in Section 5.3.

5.1 (g^-g^-) and (g^-h^-) MHV amplitudes

According to our convention $i<j$, we must have $i\in\mathsf{G}$, and $j\in\mathsf{G}$ for (g^-g^-) amplitude but $j\in\mathsf{H}$ for (g^-h^-) amplitude. To make the matrix B_{H} as simple as possible, we choose the gauge in $\{\widetilde{\epsilon}\}$ as:

$$\begin{aligned}
&(g^-g^-): &&(\widetilde{\epsilon}_a^{+})_\mu=\frac{\langle j|\overline{\sigma}_\mu|a]}{\sqrt{2}\langle ja\rangle} && && (a\in\mathsf{H})\\
&(g^-h^-): &&(\widetilde{\epsilon}_a^{+})_\mu=\frac{\langle j|\overline{\sigma}_\mu|a]}{\sqrt{2}\langle ja\rangle} && (\widetilde{\epsilon}_j^{-})_\mu=\frac{\langle j|\overline{\sigma}_\mu|1]}{\sqrt{2}[1j]} && (a\in\mathsf{H}\,,j\in\mathsf{H})\,.
\end{aligned}\tag{5.4}$$

This choice leads to $B_{\mathsf{H}}=0$ for both (g^-g^-) and (g^-h^-) amplitude. Therefore, independent of the solutions, we have:

$$\mathrm{Pf}\,(\Psi_{\mathsf{H}})=\mathrm{Pf}\begin{pmatrix}A_{\mathsf{H}} & -C_{\mathsf{H}}^T\\ C_{\mathsf{H}} & 0\end{pmatrix}=(-1)^{s(s+1)/2}\det(C_{\mathsf{H}})\,.\tag{5.5}$$

The evaluation of $\det(C_{\mathsf{H}})$ is slightly different for the two cases:

(g^-g^-) **amplitude** After we plug in the MHV solution (4.1), the matrix elements of C_{H} have the form:

$$\begin{aligned}
(C_{\mathsf{H}})_{ab}&=(F_{\theta\eta\xi})\frac{[ab]\langle bj\rangle\langle a\xi\rangle\langle b\xi\rangle}{\langle ab\rangle\langle aj\rangle} && (a,b\in\mathsf{H}\,,a\neq b)\\
(C_{\mathsf{H}})_{aa}&=-(F_{\theta\eta\xi})\langle a\xi\rangle^2\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{[ab]\langle bj\rangle\langle b\xi\rangle}{\langle ab\rangle\langle aj\rangle\langle a\xi\rangle} && (a\in\mathsf{H}\,,\text{diagonal})\,.
\end{aligned}\tag{5.6}$$

We note that in the diagonal elements $(C_{\mathsf{H}})_{aa}$, the summation is from 1 to n, *not* just within the graviton set H. After we take the determinant, the common factors can be pulled out such that:

$$\det(C_{\mathsf{H}})=(F_{\theta\eta\xi})^s\left(\prod_{a=r+1}^{n}\langle a\xi\rangle^2\right)\det(\phi_{\mathsf{H}})\,,\tag{5.7}$$

where ϕ_{H} is an $s \times s$ diagonal submatrix of the Hodges matrix ϕ:

$$\phi_{\mathsf{H}} \equiv \phi^{12\ldots r}_{12\ldots r} \qquad \{1,2\ldots r\} = \overline{\mathsf{H}} \quad (\text{complement of } \mathsf{H} \text{ in } \mathsf{N})\,. \tag{5.8}$$

Namely, ϕ_{H} is obtained from ϕ by deleting all gluon rows and columns. In this case, $\mathsf{H} = \mathsf{H}_+$ since all gravitons have positive helicities. Combining it with Eq. (5.3), we get the following expression for the $(g^- g^-)$ amplitude:

$$M_{n,r}(12\ldots r|\mathsf{H}; i^- j^-) \propto \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle r1\rangle}\det(\phi_{\mathsf{H}}) \qquad (i,j \in \mathsf{G})\,. \tag{5.9}$$

Here we have neglected an overall coefficient $(-1)^{\frac{(n-1)(n-4)+s(s+1)}{2}}$. Finally, we note that for $\mathsf{H} = \varnothing$, we just define $\det(\phi_\varnothing) = 1$ such that Eq. (5.9) returns to the Parke-Taylor formula.

$(g^- h^-)$ **amplitude** In this case, the matrix C_{H} is identical to that of Eq. (5.6), except for the $(j-r)$-th row, which is the row corresponding to the negative helicity graviton j. Moreover, since $\widetilde{\epsilon}^{\,+}_a \cdot k_j = 0$ according to our gauge choice (5.4), the only nonzero element in the column j is $(\mathsf{C}_{\mathsf{H}})_{jj}$, which equals:

$$(\mathsf{C}_{\mathsf{H}})_{jj} = \sum_{\substack{b=1\\ b\neq j}}^{n} \frac{\sqrt{2}\,\widetilde{\epsilon}^{\,-}_j \cdot k_b}{\sigma_{jb}} = (F_{\theta\eta\xi})\langle j\xi\rangle^2\,. \tag{5.10}$$

We can then expand $\det(\mathsf{C}_{\mathsf{H}})$ along the column j, which leads to:

$$\det(\mathsf{C}_{\mathsf{H}}) = (\mathsf{C}_{\mathsf{H}})_{jj}\det\left[(\mathsf{C}_{\mathsf{H}})^j_j\right] = (F_{\theta\eta\xi})^s\left(\prod_{a=r+1}^{n}\langle a\xi\rangle^2\right)\det(\phi_{\mathsf{H}_+})\,, \tag{5.11}$$

where ϕ_{H_+} is the $(s-1)\times(s-1)$ submatrix of the Hodges matrix ϕ that corresponds to positive helicity gravitons:

$$\phi_{\mathsf{H}_+} \equiv \phi^{12\ldots rj}_{12\ldots rj} \qquad \{1,2\ldots r,j\} = \overline{\mathsf{H}}_+\,. \tag{5.12}$$

After we combine it with Eq. (5.3), the final result for the $(g^- h^-)$ amplitude is:

$$M_{n,r}(12\ldots r|\mathsf{H}; i^- j^-) \propto \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle r1\rangle}\det(\phi_{\mathsf{H}_+}) \qquad (i \in \mathsf{G}, j \in \mathsf{H})\,. \tag{5.13}$$

According to Eq. (5.9) and Eq. (5.13), We can actually write a unified expression for both the $(g^- g^-)$ and $(g^- h^-)$ amplitude:

$$M_{n,r}(12\ldots r|\mathsf{H}; i^- j^-) \propto \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle r1\rangle}\det(\phi_{\mathsf{H}_+}) \qquad (i \in \mathsf{G}, j \in \mathsf{G} \text{ or } \mathsf{H})\,, \tag{5.14}$$

where for $\mathsf{H}_+ = \varnothing$, we define $\det(\phi_{\mathsf{H}_+}) = 1$.

It is interesting to note that according to Eq. (5.14), the (g^-g^-) amplitude with only two gluons must vanish identically: $M_{n,2}(1^-2^-|\mathsf{H}_+) = 0$, since the Hodges matrix only has rank $n-3$ while in this case ϕ_{H_+} is always $n-2$ dimensional. On the other hand, the (g^-h^-) amplitude with only two gluons do not vanish, since in this case ϕ_{H_+} is always $n-3$ dimensional. This pattern is correct even at 3-point:

$$\begin{aligned} M_{3,2}(1^-2^-|3^+) &\propto \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 21\rangle}\det(\phi_{\{3\}}) = 0 \\ M_{3,2}(1^-2^+|3^-) &\propto \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 21\rangle}\det(\phi_{\varnothing}) \propto \frac{\langle 13\rangle^4}{\langle 12\rangle^2}\,. \end{aligned} \tag{5.15}$$

For 3-point MHV, the Hodges matrix has rank zero since all the square brackets vanish, such that we have $\det(\phi_{\{3\}}) = \phi_{33} = 0$. This result agrees with the analysis in [82, 83].

Before moving on, we note that Selivanov [84, 85], Bern, De Freitas, and Wong [49] have given a generating function for the (g^-g^-) and (g^-h^-) amplitude (hereafter SBDW formula). We are going to prove in Section 5.3 that their results exactly agrees with Eq. (5.14), while our new expression is much simpler.

5.2 The vanishing of (h^-h^-) MHV amplitudes

In this section, we are going to show that $\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = 0$ when evaluated on the MHV solution (4.1) for the helicity configurations with $s^- \geqslant 2$. The (h^-h^-) MHV amplitude corresponds to $s^- = 2$, such that it vanishes identically.

We start with the following gauge choice:

$$(\widetilde{\epsilon}_a^-)_\mu = \frac{\langle a|\overline{\sigma}_\mu|q]}{\sqrt{2}[qa]} \quad (a \in \mathsf{H}_-)\,, \qquad (\widetilde{\epsilon}_a^+)_\mu = \frac{\langle p|\overline{\sigma}_\mu|a]}{\sqrt{2}\langle pa\rangle} \quad (a \in \mathsf{H}_+)\,, \tag{5.16}$$

where p and q are two arbitrary reference spinors, which in general are not any of the graviton momenta. After we plug in the MHV solution (4.1) into $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$, we can pull out as many common factors to rows and columns as possible, and write:

$$\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = (F_{\theta\eta\xi})^s \left(\prod_{a\in\mathsf{H}}\langle a\xi\rangle^2\right)\left(\prod_{a\in\mathsf{H}_-}\frac{\langle ap\rangle}{[aq]}\right)\mathrm{Pf}\,(\widetilde{\Psi}_{\mathsf{H}})\,, \tag{5.17}$$

where $\widetilde{\Psi}_{\mathsf{H}}$ is another $2s\times 2s$ matrix with the form:

$$\widetilde{\Psi}_{\mathsf{H}} = \begin{pmatrix} \widetilde{A} & -\widetilde{C}^T \\ \widetilde{C} & \widetilde{B} \end{pmatrix}. \tag{5.18}$$

The $s\times s$ matrices $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ have the following forms:[2]

[2] In the following, we define $\widetilde{A}_{a-r,b-r} \equiv \widetilde{A}_{\underline{ab}}$, for example.

- A-part:

$$\widetilde{A}_{\underline{ab}} = \frac{[ab]}{\langle ap\rangle\langle bp\rangle} \qquad (a, b \in \mathsf{H})\,. \tag{5.19}$$

- B-part:

$$\begin{aligned}
&\text{For } a \in \mathsf{H}_- : && \widetilde{B}_{\underline{ab}} = \begin{cases} 0 & b \in \mathsf{H}_- \\ \dfrac{\langle ap\rangle[bq]}{\langle ab\rangle} & b \in \mathsf{H}_+ \end{cases} \\
&\text{For } a \in \mathsf{H}_+ : && \widetilde{B}_{\underline{ab}} = \begin{cases} \dfrac{[aq]\langle bp\rangle}{\langle ab\rangle} & b \in \mathsf{H}_- \\ 0 & b \in \mathsf{H}_+ \end{cases}
\end{aligned} \tag{5.20}$$

- C-part:

$$\text{For } a \in \mathsf{H}_- : \quad \widetilde{C}_{\underline{ab}} = \frac{[bq]}{\langle bp\rangle}\,, \qquad \text{For } a \in \mathsf{H}_+ : \quad \widetilde{C}_{\underline{ab}} = \phi_{ab}\,. \tag{5.21}$$

In particular, those rows in H_- are identical.

Now we choose an arbitrary particle $i \in \mathsf{H}_-$ and then perform the following elementary transformations on to $\widetilde{\Psi}_{\mathsf{H}}$:

1. For all $j \in \mathsf{H}_-$ and $j \neq i$, we subtract the $(s+\underline{j})$-th row and column by the $(s+\underline{i})$-th row and column. Then in the matrix $\widetilde{C}$, all the rows corresponding to H_- are identically zero except for the $\underline{i}$-th.
2. We substract the first s rows and columns by a multiple of the $\underline{i}$-th:

$$\begin{aligned}
(\widetilde{\Psi}_{\mathsf{H}})_{\times\underline{b}} &\to (\widetilde{\Psi}_{\mathsf{H}})_{\times\underline{b}} - (\widetilde{\Psi}_{\mathsf{H}})_{\times\underline{i}}\frac{\langle ip\rangle[bq]}{[iq]\langle bp\rangle} && (b \in \mathsf{H}) \\
(\widetilde{\Psi}_{\mathsf{H}})_{\underline{a}\times} &\to (\widetilde{\Psi}_{\mathsf{H}})_{\underline{a}\times} - (\widetilde{\Psi}_{\mathsf{H}})_{\underline{i}\times}\frac{\langle ip\rangle[aq]}{[iq]\langle ap\rangle} && (a \in \mathsf{H})\,.
\end{aligned}$$

   This operation further makes the $\underline{i}$-th row of $\widetilde{C}$ zero, except for the element $\widetilde{C}_{\underline{ii}}$. Moreover, it also makes the entire matrix $\widetilde{A}$ zero, except for the $\underline{i}$-th row and column. This can be shown by using the Schouten identity:

$$\widetilde{A}_{\underline{ab}} = \frac{[ab]}{\langle ap\rangle\langle bp\rangle} \to \frac{[ab]}{\langle ap\rangle\langle bp\rangle} - \frac{[ai][bq]}{[iq]\langle ap\rangle\langle bp\rangle} - \frac{[aq][ib]}{[iq]\langle ap\rangle\langle bp\rangle} = 0\,.$$

3. Finally, we subtract the first s row and column by a multiple of the $(s+\underline{i})$-th:

$$\begin{aligned}
(\widetilde{\Psi}_{\mathsf{H}})_{\times\underline{b}} &\to (\widetilde{\Psi}_{\mathsf{H}})_{\times\underline{b}} - (\widetilde{\Psi}_{\mathsf{H}})_{\times,s+\underline{i}}\frac{[bi]}{[iq]\langle bp\rangle} && (b \in \mathsf{H}) \\
(\widetilde{\Psi}_{\mathsf{H}})_{\underline{a}\times} &\to (\widetilde{\Psi}_{\mathsf{H}})_{\underline{a}\times} - (\widetilde{\Psi}_{\mathsf{H}})_{s+\underline{i},\times}\frac{[ai]}{[iq]\langle ap\rangle} && (a \in \mathsf{H})\,,
\end{aligned}$$

such that the entire matrix $\widetilde{A}$ is made zero. Then for those rows and columns corresponding to H_+, we perform:

$$
\begin{aligned}
(\widetilde{\Psi}_{\mathsf{H}})_{\times,s+\underline{b}} &\to (\widetilde{\Psi}_{\mathsf{H}})_{\times,s+\underline{b}} - (\widetilde{\Psi}_{\mathsf{H}})_{\times,s+\underline{i}}\frac{\langle ip\rangle[bi]}{[iq]\langle bi\rangle} && b\in\mathsf{H}_+ \\
(\widetilde{\Psi}_{\mathsf{H}})_{s+\underline{a},\times} &\to (\widetilde{\Psi}_{\mathsf{H}})_{s+\underline{a},\times} - (\widetilde{\Psi}_{\mathsf{H}})_{s+\underline{i},\times}\frac{\langle ip\rangle[ai]}{[iq]\langle ai\rangle} && a\in\mathsf{H}_+\,,
\end{aligned}
$$

such that the only nonzero element in the $\underline{i}$-th row and column is $\widetilde{C}_{\underline{ii}} = [iq]/\langle ip\rangle$. Now we can just pull out $\widetilde{C}_{\underline{ii}}$ and write:

$$
\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = (-1)^s\frac{[iq]}{\langle ip\rangle}\mathrm{Pf}\,(\widetilde{\Psi}'_{\mathsf{H}})\,, \tag{5.22}
$$

where the $(2s-2)\times(2s-2)$ matrix $\widetilde{\Psi}'_{\mathsf{H}}$ has the shape:

$$
\widetilde{\Psi}'_{\mathsf{H}} = \begin{array}{c|c|c|}
 & \overleftrightarrow{s+s^- -2} & \\
\hline
s+s^- -2 & 0 & -\widetilde{C}^T_+ \text{ part} \\
\hline
s^+ & \widetilde{C}_+ \text{ part} & \widetilde{B} \text{ part} \\
\hline
\end{array}\,. \tag{5.23}
$$

The exact form of each matrix element here is not important. The lower left $\widetilde{C}_+$ block has dimension $s^+\times(s+s^- -2)$, such that its columns are more than the rows if $s^-\geqslant 2$. For these cases, we can always find an elemetary transformation to make at least one entire row and column zero. As a result, we have proved that $\mathrm{Pf}\,(\Psi_{\mathsf{H}})=0$ for $s^-\geqslant 2$ when evaluated on the MHV solution (4.1). In particular, the (h^-h^-) MHV amplitude is identically zero.

Moreover, we will show in Chapter 6 that using the similar technique, we can prove that *if gluons have the same helicity, all the single trace tree level EYM amplitudes must vanish, independent of the graviton helicity configurations.* This statement is first conjectured in [49].

5.3 Spanning forests and MHV amplitudes

Using the CHY formalism for EYM, we have derived in the previous section a set of very simple and compact expressions for single trace tree level MHV amplitudes:

$$
M_{n,r}(12\ldots r|\mathsf{H};i^-j^-) = \begin{cases} \dfrac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle r1\rangle}\det(\phi_{\mathsf{H}_+}) & (i\in\mathsf{G}\,,j\in\mathsf{G}\text{ or }\mathsf{H}) \\ 0 & (i,j\in\mathsf{H}) \end{cases}\,. \tag{5.24}
$$

On the other hand, the SBDW formula [49, 84, 85] states that the (g^-g^-) and (g^-h^-) MHV amplitude have the following form:

$$M_{n,r}(12\ldots r|\mathsf{H};i^-j^-) \propto (-1)^{s^+}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle r1\rangle}\mathcal{S}(\mathsf{H}_+;\mathsf{N}) \qquad (i\in\mathsf{G}\,,j\in\mathsf{G}\text{ or }\mathsf{H})\,, \tag{5.25}$$

where the graviton factor $\mathcal{S}(\mathsf{H}_+)$ can be obtained from a generating function:[3]

$$\begin{aligned}&G(a_\mu\,,\mathsf{H}_+;\mathsf{N}) \qquad\qquad (\mu\in\mathsf{H}_+\text{ and }\overline{H}_+=\text{the set complement of }\mathsf{H}_+\text{ in }\mathsf{N})\\ &=\exp\left\{\sum_{n_1\in\mathsf{H}_+}a_{n_1}\sum_{l\in\overline{\mathsf{H}}_+}\psi_{n_1l}\exp\left[\sum_{\substack{n_2\in\mathsf{H}_+\\ n_2\neq n_1}}a_{n_2}\psi_{n_2n_1}\exp\left(\sum_{\substack{n_3\in\mathsf{H}_+\\ n_3\neq n_1\,,n_2}}a_{n_3}\psi_{n_3n_2}\exp(\ldots)\right)\right]\right\}\end{aligned} \tag{5.26}$$

in which the matrix element ψ_{ab} is defined as:

$$\psi_{ab}=\phi_{ab}\frac{\langle b\xi\rangle\langle b\eta\rangle}{\langle a\xi\rangle\langle a\eta\rangle}\,. \tag{5.27}$$

Here ξ and η are the two reference spinors used in the diagonal elements ϕ_{aa}:

$$\phi_{aa}=-\sum_{\substack{l=1\\ l\neq a}}^{n}\frac{[al]\langle l\xi\rangle\langle l\eta\rangle}{\langle al\rangle\langle a\xi\rangle\langle a\eta\rangle}\,.$$

Then $\mathcal{S}(\mathsf{H}_+;\mathsf{N})$ is just a certain Taylor expansion coefficient of $G(a_\mu\,,\mathsf{H}_+;\mathsf{N})$:

$$\mathcal{S}(\mathsf{H}_+;\mathsf{N})=\left(\prod_{m\in\mathsf{H}_+}\frac{\partial}{\partial a_m}\right)G(a_\mu\,,\mathsf{H}_+;\mathsf{N})\Bigg|_{a_m=0}\,. \tag{5.28}$$

The SBDW formula looks very different from what we have obtained from CHY at the first glance. In this section, we are going to prove that they are exactly equivalent, namely,

$$\mathcal{S}(\mathsf{H}_+;\mathsf{N})=(-1)^{s^+}\det(\phi_{\mathsf{H}_+})\,, \tag{5.29}$$

using a graph theory technique.

[3] In [49], the second summation is over $l\in\mathsf{G}_+$ instead of $l\in\overline{\mathsf{H}}_+$. The reason is that [49] has used a special gauge choice $\xi=i$ and $\eta=j$ as in Eq. (5.27). The final amplitude is of course gauge invariant so that our definition is equivalent to that of [49].

5.3.1 Generating the spanning forests

We can construct a weighted complete graph K_n with the vertex set $V_n = \{v_1, v_2 \ldots v_n\}$, and ψ_{ab} as the weight on the edge $v_a v_b$.[4] The information of such a graph K_n is encoded in the $n \times n$ *Laplace matrix* W_n, defined as:

$$(W_n)_{ab} = \begin{cases} -\psi_{ab} & a \neq b \\ \sum\limits_{b=1\,, b\neq a}^{n} \psi_{ab} & \text{diagonal} \end{cases} . \tag{5.30}$$

The matrix W_n is useful since all its diagonal minors must equal that of ϕ. In particular, we have:

$$\det\left[(W_n)_{\mathsf{H}_+}\right] = \frac{\prod_{b\in\mathsf{H}_+}\langle b\xi\rangle\langle b\eta\rangle}{\prod_{a\in\mathsf{H}_+}\langle a\xi\rangle\langle a\eta\rangle}(-1)^{s^+}\det(\phi_{\mathsf{H}_+}) = (-1)^{s^+}\det(\phi_{\mathsf{H}_+})\,. \tag{5.31}$$

Therefore, to prove Eq. (5.29), we can equivalently show that:

$$\mathcal{S}(\mathsf{H}_+;\mathsf{N}) = \det\left[(W_n)_{\mathsf{H}_+}\right]. \tag{5.32}$$

The benefit of studying W_n is that now we can use the graph theory to our favor. First, we have a graph combinatorical interpretation for $\mathcal{S}$, according to the following theorem:

Theorem 5.1: *Suppose $I_r = \{i_1, i_2 \ldots i_r\}$ is an r-element subset of the vertex set V_n, then the function $G(a_\mu\,, \bar{I}_r;\mathsf{N})$ is the generating function of all the weighted spanning forests of K_n rooted in I_r in the following sense:*

$$\mathcal{S}(\bar{I}_r;\mathsf{N}) = \left(\prod_{m\in\bar{I}_r}\frac{\partial}{\partial a_m}\right)G(a_\mu\,,\bar{I}_r;\mathsf{N})\bigg|_{a_m=0} = \sum_{F\in\mathcal{F}_{I_r}(K_n)}\left(\prod_{v_a v_b\in E(F)}\psi_{ab}\right), \tag{5.33}$$

where $\mathcal{F}_{I_r}(K_n)$ denotes the set of the spanning forests of K_n rooted in I_r. The edge set $E(F)$ of a forest F consists of all edges $v_a v_b$ that are directed into the roots.

Several spanning forest examples are given in Figure 5.1. We will give another 7-point example in Section 5.3.2 to demonstrate the relation (5.33). Then we will give the general proof in Section 5.3.3.

---

[4] If ψ_{ab} is symmetric, the graph is undirected. Otherwise, the graph is *directed* (our case).

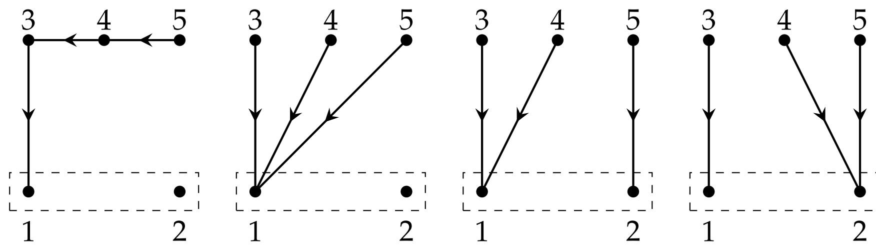


Figure 5.1. Some spanning forest examples. Here we show several spanning forests of K_5 rooted in $I_2 = \{1, 2\}$ (the vertices enclosed by the dashed boxes).

On the other hand, the evaluation of $\det\left[(W_n)_{\overline{I}_r}\right]$ has the same graph combinatorical interpretation [86]:[5]

$$\det\left[(W_n)_{\overline{I}_r}\right] = \sum_{F \in \mathcal{F}_{I_r}(K_n)} \left(\prod_{v_a v_b \in E(F)} \psi_{ab} \right) . \tag{5.34}$$

Therefore, according to Theroem 5.1, we have established the identity:

$$\mathcal{S}(\overline{I}_r; \mathsf{N}) = \det\left[(W_n)_{\overline{I}_r}\right] . \tag{5.35}$$

Then after we choose $\overline{I}_r = \mathsf{H}_+$, the desired relation (5.32) follows immediately.

In this way, we have proved that our new expression Eq. (5.14) is equivalent to the SBDW formula. However, our expression is much simpler and easier to evaluate, as the example in Section 5.3.2 will demonstrate.

5.3.2 A seven-point example

Now we demonstrate that the graviton factor of the 7-point amplitude:

$$M_{7,4}(1_g^- 2_g^- 3_g^+ 4_g^+ | 5_h^+ 6_h^+ 7_h^+)$$

indeed satisfies the relation (5.33). The root set is $\overline{\mathsf{H}}_+ = \{1, 2, 3, 4\}$. However, we can choose the reference spinors in Eq. (5.27) as $\xi = 1$ and $\eta = 2$, such that:

$$\psi_{a1} = \psi_{a2} = 0 \qquad \text{for all } a \in \{3, 4, 5, 6, 7\} .$$

In other words, the root 1 and 2 do not connect to any other vertices. The problem is thus reduced to summing over all the spanning forests with root set $\{3, 4\}$ of the graph K_5 whose vertex set is $V_5 = \{3, 4, 5, 6, 7\}$. The SBDW generating function is:

[5]This relation is called matrix-tree theorem II in [86].

$$G(a_5, a_6, a_7; V_5) = \exp\Big\{a_5(\psi_{53} + \psi_{54}) \exp\left[a_6\psi_{65}\exp(a_7\psi_{76}) + a_7\psi_{75}\exp(a_6\psi_{67})\right]$$
$$+ a_6(\psi_{63} + \psi_{64}) \exp\left[a_7\psi_{76}\exp(a_5\psi_{57}) + a_5\psi_{56}\exp(a_7\psi_{75})\right]$$
$$+ a_7(\psi_{73} + \psi_{74}) \exp\left[a_5\psi_{57}\exp(a_6\psi_{65}) + a_6\psi_{67}\exp(a_5\psi_{56})\right]\Big\}.$$

Our main task here is to verify that the graviton factor

$$\mathcal{S}(\{5,6,7\}; V_5) = \frac{\partial}{\partial a_5}\frac{\partial}{\partial a_6}\frac{\partial}{\partial a_7} G(a_5, a_6, a_7; V_5)\Big|_{a_5=a_6=a_7=0} \tag{5.36}$$

indeed gives the graph combinatorical result

$$\sum_{F\in\mathcal{F}_{\{3,4\}}(K_5)} \left(\prod_{v_a v_b \in E(F)} \psi_{ab}\right). \tag{5.37}$$

Our strategy is to expand both Eq. (5.36) and Eq. (5.37) with respect to ψ_{l3} and then compare them order by order.

First, Eq. (5.37) can be expanded as:

$$\sum_{F\in\mathcal{F}_{\{3,4\}}(K_5)} \left(\prod_{v_a v_b \in E(F)} \psi_{ab}\right) = A + \sum_{l\in\{5,6,7\}} \psi_{l3} B_l + \sum_{l,k\in\{5,6,7\}} \psi_{l3}\psi_{k3} C_{lk} + \psi_{53}\psi_{63}\psi_{73} D. \tag{5.38}$$

Effectively, this expansion puts all the spanning forests into four categories $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$. In category $\boldsymbol{A}$, which contributes only to A, ψ_{l3} does not appear, which means that all the leaves are grown from the root 4. Therefore, A is contributed by the spanning trees of $V_4 = \{4, 5, 6, 7\}$ with root 4. There are in all 16 of them, as shown in Figure 5.2. Similarly in category $\boldsymbol{B}$, the coefficient B_5 is contributed by those forests in which if the leaves $\{6, 7\}$ belong to the root 3, they must first converge to 5. In other words, B_5 is contributed by all the spanning forests of $V_4 = \{4, 5, 6, 7\}$ with root set $\{4, 5\}$. In general, each category corresponds to a set of spanning forests of $V_4 = \{4, 5, 6, 7\}$ with different root sets, which is collected in Table 5.1. Therefore, each of the coefficients corresponds to one of the sub-problems with one less vertex. This feature strongly suggests an inductive proof for the general relation Eq. (5.33).

Next, we need to verify that the Taylor expansion of $\mathcal{S}(\{5, 6, 7\}; V_5)$ with respect to ψ_{l3} exactly reproduces the same terms as in Eq. (5.38). To start with, the zeroth order can be simply obtained by setting all $\psi_{l3} = 0$ in the generating function:

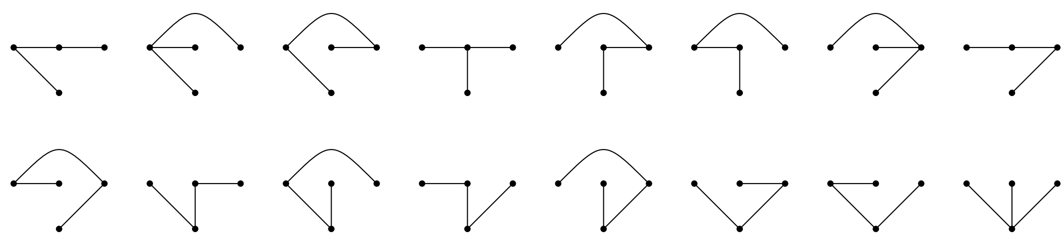


Figure 5.2. The 16 spanning trees of K_4. The direction on each edge is understood as pointing towards the root, which is the bottom vertex.

Table 5.1. The corresponding spanning forests of $V_4 = \{4, 5, 6, 7\}$. They are separated into category $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ according to Eq. (5.38).

category	root set	# of diagrams
A	$\{4\}$	16
B_l	$\{4, l\}$	$8 \times 3 = 24$
C_{lk}	$\{4, l, k\}$	$3 \times 3 = 9$
D	$\{4, 5, 6, 7\}$	1

$$
\begin{aligned}
G(a_5, a_6, a_7; V_4) = \; & G(a_5, a_6, a_7; V_5)|_{\psi_{53}=\psi_{63}=\psi_{73}=0} \\
= \; & \exp\Big\{ a_5\psi_{54} \exp\left[a_6\psi_{65}\exp(a_7\psi_{76}) + a_7\psi_{75}\exp(a_6\psi_{67})\right] \\
& + a_6\psi_{64}\exp\left[a_7\psi_{76}\exp(a_5\psi_{57}) + a_5\psi_{56}\exp(a_7\psi_{75})\right] \\
& + a_7\psi_{74}\exp\left[a_5\psi_{57}\exp(a_6\psi_{65}) + a_6\psi_{67}\exp(a_5\psi_{56})\right] \Big\}.
\end{aligned}
$$

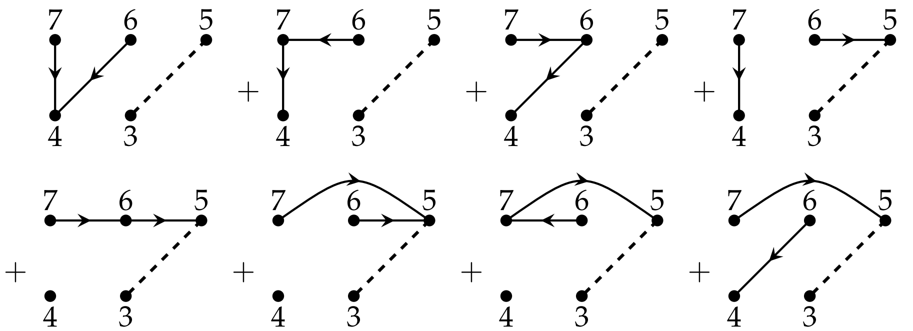
By a straightforward but tedious calculation, we obtain:

$$
A = \mathcal{S}(\{5,6,7\}; V_4) = \frac{\partial}{\partial a_5}\frac{\partial}{\partial a_6}\frac{\partial}{\partial a_7} G(a_5, a_6, a_7\}; V_4)\bigg|_{a_5=a_6=a_7=0}, \tag{5.39}
$$

namely, $\mathcal{S}(\{5,6,7\}; V_4)$ contains 16 terms that exactly correspond to the 16 spanning trees as shown in Figure 5.2.

For the coefficients in category $\boldsymbol{B}$, it is sufficient to calculate just B_5 as an example. On the graph theory side, we have:

$$B_5 =$$



$$= \psi_{74}\psi_{64} + \psi_{67}\psi_{74} + \psi_{76}\psi_{64} + \psi_{74}\psi_{65} + \psi_{76}\psi_{65} + \psi_{75}\psi_{65} + \psi_{67}\psi_{75} + \psi_{75}\psi_{64}\,. \tag{5.40}$$

According to Table 5.1, the solid edges just consist of the spanning forests of V_4 with the root set $\{4, 5\}$. In $G(a_5, a_6, a_7; V_5)$, we find that ψ_{53} only appears in the outmost level of the exponents. Then the Taylor expansion coefficient of ψ_{53} can be extracted by acting $\partial/\partial a_5$ only on $a_5\psi_{53}$, and then set $\psi_{63} = \psi_{73} = 0$.[6] The result is:

$$G(a_6, a_7; V_4) = \exp\left[a_6(\psi_{64} + \psi_{65})\exp(a_7\psi_{76}) + a_7(\psi_{74} + \psi_{75})\exp(a_6\psi_{67})\right]\,. \tag{5.41}$$

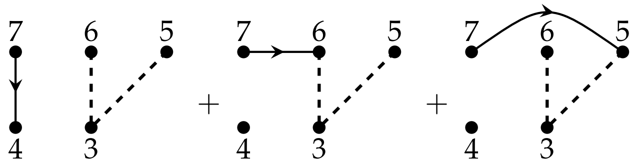
Now we can explicitly verify that:

$$\begin{aligned}
B_5 = \mathcal{S}(\{6,7\}; V_4) = {}& \frac{\partial}{\partial a_6}\frac{\partial}{\partial a_7} G(a_6, a_7; V_4)\bigg|_{a_6=a_7=0} \\
= {}& \frac{\partial}{\partial a_6}\left[(\psi_{74} + \psi_{75})e^{a_6\psi_{67}} + a_6(\psi_{64} + \psi_{65})\psi_{76}\right] e^{a_6(\psi_{64}+\psi_{65})}\bigg|_{a_6=0} \\
= {}& (\psi_{74} + \psi_{75})\psi_{67} + (\psi_{64} + \psi_{65})\psi_{76} + (\psi_{64} + \psi_{65})(\psi_{74} + \psi_{75})\,.
\end{aligned} \tag{5.42}$$

The coefficient B_6 and B_7 can also be calculated in this way, and they do agree with a graph summation similar to Eq. (5.40).

Then for the coefficients in category **C**, again we only calculate C_{56} as an example. On the graph theory side, we have the spanning forests of V_4 with the root set $\{4, 5, 6\}$:

$$C_{56} =$$



$$= \psi_{74} + \psi_{75} + \psi_{76}\,. \tag{5.43}$$

---

[6]The other ways of acting the derivative give only lower order terms in ψ_{53}. We set $\psi_{63} = \psi_{73} = 0$ since they only contribute to higher order terms in the ψ_{l3} expansion.

In the graviton factor $\mathcal{S}(\{5,6,7\};V_5)$, the coefficient of $\psi_{53}\psi_{63}$ is generated by $G(a_7;V_4)$, which is obtained from $G(a_5,a_6,a_7;V_5)$ by acting the derivative $(\partial/\partial a_5)(\partial/\partial a_6)$ on to $a_5\psi_{53}$ and $a_6\psi_{63}$ only, and then setting $a_5 = a_6 = \psi_{73} = 0$:

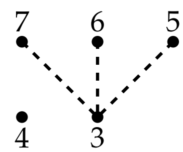
$$G(a_7;V_4) = \exp\left[a_7(\psi_{74}+\psi_{75}+\psi_{76})\right] . \tag{5.44}$$

Indeed, the two coefficients match:

$$C_{56} = \mathcal{S}(\{7\};V_4) = \frac{\partial}{\partial a_7} G(a_7;V_4)\bigg|_{a_7=0} = \psi_{74}+\psi_{75}+\psi_{76} . \tag{5.45}$$

Finally, the category $\boldsymbol{D}$ only contains one trivial graph:

$D =$



$$= 1 . \tag{5.46}$$

On the other hand, by acting all the three derivatives onto $a_5\psi_{53}$, $a_6\psi_{63}$, and $a_7\psi_{73}$, we get the same result:

$$D = \mathcal{S}(\varnothing;V_4) = G(\varnothing;V_4) = 1 . \tag{5.47}$$

Now we have checked that each term in the Taylor expansion of $\mathcal{S}(\{5,6,7\};V_5)$:

$$\begin{aligned} \mathcal{S}(\{5,6,7\};V_5) = \mathcal{S}(\{5,6,7\};V_4) &+ \sum_{l\in\{5,6,7\}} \psi_{l3}\,\mathcal{S}(\{5,6,7\}\backslash\{l\};V_4) \\ &+ \sum_{l,k\in\{5,6,7\}} \psi_{l3}\psi_{k3}\,\mathcal{S}(\{5,6,7\}\backslash\{l,k\};V_4) + \psi_{53}\psi_{63}\psi_{73}\,\mathcal{S}(\varnothing;V_4) \end{aligned} \tag{5.48}$$

exactly matches that of the graph expansion Eq. (5.38), such that the desired equality holds:

$$\mathcal{S}(\{5,6,7\};V_5) = \sum_{F\in\mathcal{F}_{\{3,4\}}(K_5)} \left(\prod_{v_a v_b \in E(F)} \psi_{ab} \right) . \tag{5.49}$$

However, when performing this calculation, we are encountered by the summation over:

$$|\boldsymbol{A}| + |\boldsymbol{B}| + |\boldsymbol{C}| + |\boldsymbol{D}| = 16 + 8\times 3 + 3\times 3 + 1 = 50$$

terms generated by different ways of acting derivatives. Therefore, the SBDW style evaluation is very calculationally heavy, which will quickly grow out of control for more points. In fact, those 50 terms can be nicely grouped into 3×3 determinant, as our new formula Eq. (5.14) makes manifest. This example also demonstrates that the CHY formalism is superior than the SBDW generating function in evaluating the MHV amplitudes for EYM.

5.3.3 General relation

Gaining enough experience from the explicit 7-point example, we are now ready to present the general proof of Theorem 5.1. First, at $n = 2$, we have only two vertices. We can just choose the root to be $I = \{1\}$ such that $\overline{I} = \{2\}$. Then we can show that

$$\begin{tikzcd} \bullet\, v_2 \arrow[d, "\psi_{21}"'] \\ \bullet\, v_1 \end{tikzcd} = \psi_{21} = S(\{2\};\{1,2\}) = \frac{d}{da_2}\exp\left(a_2\psi_{21}\right)\Big|_{a_2=0} . \tag{5.50}$$

such that Eq. (5.33) holds at $n = 2$. Next, we assume that Eq. (5.33) holds at $(n-1)$-point and check whether it also holds at n-point. For a forest F with roots $I_r = \{i_1, i_2 \ldots i_r\}$, we can classify it by the set of vertices that are immediately connected to the root i_r. Suppose the vertex set $P_t = \{p_1, p_2 \ldots p_t\} \subset \overline{I}_r$ is directly connected to i_r, then we have:

$$\prod_{v_a v_b \in E(F)} \psi_{ab} = \left(\prod_{k=1}^{t} \psi_{p_k i_r}\right) \prod_{v_a v_b \in E(\widetilde{F})} \psi_{ab} \qquad \widetilde{F} \in \mathcal{F}_{I_{r-1+t}}(K_{n-1}) , \tag{5.51}$$

where $\widetilde{F}$ is an $(n-1)$-point forest with roots $I_{r-1+t} = \{i_1, i_2 \ldots i_{r-1}, p_1, p_2 \ldots p_t\}$. This expansion is depicted in Figure 5.3. Therefore, the right hand side of Eq. (5.33) can be expanded as:

$$\begin{aligned} \sum_{F \in \mathcal{F}_{I_r}(K_n)} \left(\prod_{v_a v_b \in E(F)} \psi_{ab}\right) &= \sum_{t=0}^{n-r} \sum_{P_t \subset \overline{I}_r} \left(\prod_{k=1}^{t} \psi_{p_k i_r}\right) \sum_{\widetilde{F} \in \mathcal{F}_{I_{r-1+t}}(K_{n-1})} \left(\prod_{v_a v_b \in E(\widetilde{F})} \psi_{ab}\right) \\ &= \mathcal{S}(\overline{I}'_{r-1}; \mathsf{N}') + \sum_{t=1}^{n-r} \sum_{P_t \subset \overline{I}_r} \left(\prod_{k=1}^{t} \psi_{p_k i_r}\right) \mathcal{S}(\overline{I}'_{r-1+t}; \mathsf{N}') , \end{aligned} \tag{5.52}$$

where $\mathsf{N}' = \mathsf{N} \backslash \{i_r\}$ and $\overline{I}\,'$ is defined as the complement of I in the set N'. Consequently, we must have $\overline{I}_r = \overline{I}'_{r-1}$. We note that the first line is nothing but the generalization of Eq. (5.38) in our example, and the second line is obtained from the induction assumption.

Our next job is to verify that Eq. (5.52) agrees with the Taylor expansion of $\mathcal{S}(\overline{I}_r; \mathsf{N})$ with respect to $\psi_{p_k i_r}$. If this is the case, then we just prove that Eq. (5.33) holds at n-point, which completes the inductive proof of Eq. (5.33) and thus Theorem 5.1. To start with, the zeroth order of $\mathcal{S}(\overline{I}_r; \mathsf{N})$ can be extracted by setting all $\psi_{p_k i_r} = 0$ in the generating function $G(a_\mu, \overline{I}_r; \mathsf{N})$. According to Eq. (5.26), all $\psi_{p_k i_r}$ appear only in the outmost level of exponent in $G(a_\mu, \overline{I}_r; \mathsf{N})$ in the form:

$$\sum_{n_1 \in \overline{I}_r} a_{n_1} \sum_{l \in I_r} \psi_{n_1 l} = \sum_{n_1 \in \overline{I}'_{r-1}} a_{n_1} \left(\psi_{n_1 i_r} + \sum_{l \in I_{r-1}} \psi_{n_1 l}\right) . \tag{5.53}$$

F: solid and dashed edges
$\widetilde{F}$: solid edges only

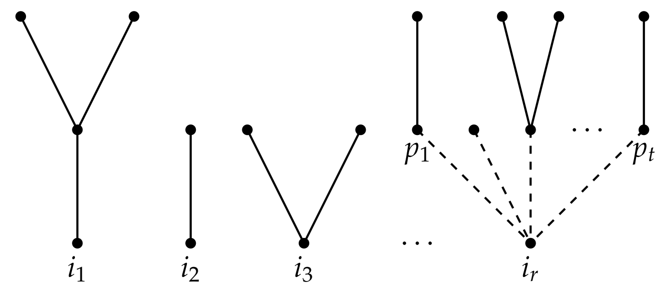


Figure 5.3. A classification of spanning forests. Here, we show that an n-point forest rooted in the set $\{i_1, i_2 \dots i_r\}$ can be constructed from an $(n-1)$-point forest rooted in $\{i_1, i_2 \dots i_{r-1}, p_1 \dots p_t\}$, with $\{p_1 \dots p_t\}$ connected to i_r.

Then by setting all $\psi_{n_1 i_r}$ to zero, the exponent becomes:

$$\left[\sum_{n_1 \in \overline{I}_r} a_{n_1} \sum_{l \in I_r} \psi_{n_1 l} \exp(\dots)\right]_{\psi_{n_1 i_r}=0} = \sum_{n_1 \in \overline{I}'_{r-1}} a_{n_1} \sum_{l \in I_{r-1}} \psi_{n_1 l} \exp(\dots)\,, \tag{5.54}$$

such that $G(a_\mu, \overline{I}_r; \mathsf{N})$ reduces to:

$$\begin{aligned} G(a_\mu, \overline{I}_r; \mathsf{N})\big|_{\psi_{n_1 i_r}=0} &= \exp\left[\sum_{n_1 \in \overline{I}'_{r-1}} a_{n_1} \sum_{l \in I_{r-1}} \psi_{n_1 l} \exp\left(\sum_{\substack{n_2 \in \overline{I}'_{r-1} \\ n_2 \neq n_1}} a_{n_2} \psi_{n_2 n_1} \exp(\dots)\right)\right] \\ &= G(a_\mu, \overline{I}'_{r-1}; \mathsf{N}')\,. \end{aligned} \tag{5.55}$$

Therefore, the zeroth order in $\mathcal{S}(\overline{I}_r; \mathsf{N})$ agrees with the first term of Eq. (5.52):

$$\mathcal{O}(1): \qquad \left(\prod_{m \in \overline{I}'_{r-1}} \frac{\partial}{\partial a_m}\right) G(a_\mu, \overline{I}'_{r-1}; \mathsf{N}')\Bigg|_{a_m=0} = \mathcal{S}(\overline{I}'_{r-1}; \mathsf{N}')\,. \tag{5.56}$$

Next, we go to a generic order t with fixed $P_t = \{p_1, p_2 \dots p_t\}$, and study the Taylor expansion at the order $\mathcal{O}(\psi_{p_1 i_r} \dots \psi_{p_t i_r})$. In this case, the derivative part can be separated into two groups:

$$\prod_{m \in \overline{I}_r} \frac{\partial}{\partial a_m} = \left(\prod_{m \in \overline{I}'_{r-1+t}} \frac{\partial}{\partial a_m}\right) \underline{\left(\prod_{k=1}^{t} \frac{\partial}{\partial a_{p_k}}\right)}\,. \tag{5.57}$$

Now we can put the outmost exponent into four groups:

$$\begin{aligned} \sum_{n_1 \in \overline{I}_r} a_{n_1} \sum_{l \in I_r} \psi_{n_1 l} = &\sum_{n_1 \in \overline{I}'_{r-1+t}} a_{n_1} \sum_{l \in I_{r-1}} \psi_{n_1 l} + \sum_{n_1 \in \overline{I}'_{r-1+t}} a_{n_1} \psi_{n_1 i_r} \\ &+ \sum_{k=1}^{t} a_{p_k} \sum_{l \in I_{r-1}} \psi_{p_k l} + \underline{\sum_{k=1}^{t} a_{p_k} \psi_{p_k i_r}}\,. \end{aligned} \tag{5.58}$$

We can first set all $\psi_{n_1 i_r} = 0$ since they do not contribute to the order $\mathcal{O}(\psi_{p_1 i_r} \dots \psi_{p_t i_r})$ currently under consideration. Then when acting the underlined derivatives in Eq. (5.57) onto $G(a_\mu, \bar{I}_r; \mathsf{N})$, we will reproduce the desired $\mathcal{O}(\psi_{p_1 i_r} \dots \psi_{p_t i_r})$ term only if all of them are acted on the underlined term in Eq. (5.58). All the other ways of distributing the derivatives will result in less $\psi_{p_k i_r}$ than required, so that they give lower order terms in the expansion. In this sense, we can then define an "effective" generating function by setting all the other $a_{p_k} = 0$ except for the underlined ones in Eq. (5.58):

$$\begin{aligned} G^{\text{eff}}(a_\mu, \bar{I}_r; \mathsf{N}) = \exp\Bigg[& \sum_{n_1 \in \bar{I}'_{r-1+t}} a_{n_1} \sum_{l \in I_{r-1}} \psi_{n_1 l} \exp\Bigg(\sum_{\substack{n_2 \in \bar{I}'_{r-1+t} \\ n_2 \neq n_1}} a_{n_2} \psi_{n_2 n_1} \exp(\dots) \Bigg) \\ & + \sum_{k=1}^{t} a_{p_k} \psi_{p_k i_r} \exp\Bigg(\sum_{n_2 \in \bar{I}'_{r-1+t}} a_{n_2} \psi_{n_2 p_k} \exp(\dots) \Bigg) \Bigg], \end{aligned} \tag{5.59}$$

where in the higher level exponents, we have used, for example:

$$\begin{aligned} \sum_{\substack{n_2 \in \bar{I}_r \\ n_2 \neq n_1}} a_{n_2} \psi_{n_2 n_1} \exp(\dots) \Bigg|_{a_{p_k}=0} &= \sum_{\substack{n_2 \in \bar{I}'_{r-1+t} \\ n_2 \neq n_1}} a_{n_2} \psi_{n_2 n_1} \exp(\dots) \\ \sum_{\substack{n_2 \in \bar{I}_r \\ n_2 \neq n_1}} a_{n_2} \psi_{n_2 p_k} \exp(\dots) \Bigg|_{a_{p_k}=0} &= \sum_{n_2 \in \bar{I}'_{r-1+t}} a_{n_2} \psi_{n_2 p_k} \exp(\dots) . \end{aligned}$$

After acting the underlined derivatives in Eq. (5.57) onto this G^{eff}, we get nothing but the generating function $G(a_\mu, \bar{I}'_{r-1+t}; \mathsf{N}')$:

$$\begin{aligned} & \left(\prod_{k=1}^{t} \frac{\partial}{\partial a_{p_k}} \right) G^{\text{eff}}(a_\mu, \bar{I}_r; \mathsf{N}) \\ & = \left(\prod_{k=1}^{t} \psi_{p_k i_r} \right) \exp\left[\sum_{n_2 \in \bar{I}'_{r-1+t}} a_{n_2} \sum_{k=1}^{t} \psi_{n_2 p_k} \exp(\dots) \right] \exp\left[\sum_{n_1 \in \bar{I}'_{r-1+t}} a_{n_1} \sum_{l \in I_{r-1}} \psi_{n_1 l} \exp(\dots) \right] \\ & = \left(\prod_{k=1}^{t} \psi_{p_k i_r} \right) \exp\left[\sum_{n_2 \in \bar{I}'_{r-1+t}} a_{n_1} \sum_{l \in I_{r-1+t}} \psi_{n_2 l} \exp(\dots) \right] \\ & = \left(\prod_{k=1}^{t} \psi_{p_k i_r} \right) G(a_\mu, \bar{I}'_{r-1+t}; \mathsf{N}') . \end{aligned} \tag{5.60}$$

To obtain the third line, we have renamed the index n_2 to n_1, n_3 to n_2, etc, in the first exponential factor. Therefore, at the order $\mathcal{O}(\psi_{p_1 i_r} \dots \psi_{p_t i_r})$, the Taylor expansion of $\mathcal{S}(\bar{I}_r; \mathsf{N})$ agrees with that of Eq. (5.52):

$$\mathcal{O}(\psi_{p_1 i_r} \ldots \psi_{p_t i_r}) : \qquad \left(\prod_{m \in \bar{I}'_{r-1+t}} \frac{\partial}{\partial a_m} \right) G(a_\mu, \bar{I}'_{r-1+t}; \mathsf{N}') = \mathcal{S}(\bar{I}'_{r-1+t}; \mathsf{N}') \,. \tag{5.61}$$

Since our choice of $P_t = \{p_1, p_2 \ldots p_t\}$ is completely general, we have just proved that $\mathcal{S}(\bar{I}_r; \mathsf{N})$ has the Taylor expansion Eq. (5.52):

$$\mathcal{S}(\bar{I}_r; \mathsf{N}) = \mathcal{S}(\bar{I}'_{r-1}; \mathsf{N}') + \sum_{t=1}^{n-r} \sum_{P_t \subset \bar{I}_r} \left(\prod_{k=1}^{t} \psi_{p_k i_r} \right) \mathcal{S}(\bar{I}'_{r-1+t}; \mathsf{N}') \,. \tag{5.62}$$

This completes our inductive proof of Eq. (5.33) and thus Theorem 5.1.

CHAPTER 6

SOLUTIONS AND HELICITY CONFIGURATIONS

We have already seen in Chapter 4 that the MHV solution Eq. (4.1) alone can reproduce the correct Yang-Mills and gravity MHV amplitudes in four dimensions. We have also suggested in Section 4.4 that all the other solutions do not contribute to MHV amplitudes since they all make $\mathrm{rank}(C_+) < n-3$ at MHV. In this chapter, we are going to address a more general problem: how to characterize those solutions that contribute to the N^{k}MHV amplitudes. Actually, the study in Chapter 4 has already given us a hint: we can use the rank of some matrix. Moreover, this matrix must be closely related to the C matrix defined in Eq. (3.31).

The outline of this chapter is as follows. In Section 6.1, we define two discriminant matrices $\mathfrak{C}_\pm$ and give a general argument that their ranks can link the solutions to SE to helicity configurations. The details on how to characterize the solutions are then presented in Section 6.2. Next, we prove that those solutions that belong to the sector k in this characterization can only support the N^{k}MHV Yang-Mills and gravity amplitudes, not any other $k' \neq k$, in Section 6.4. Finally, we give some applications of this technique to single trace EYM amplitudes in Section 6.5. In particular, we can prove from the CHY's perspective that if gluons have the same helicity, then the single trace EYM amplitudes must vanish, independent of the graviton helicities. Many of these results can be found in our work [41].

6.1 Discriminant matrices

We find that the ranks of the following two $n \times n$ matrices can be used to characterize the solutions to SE:

$$(\mathfrak{C}_-)_{ab} = \begin{cases} \dfrac{\langle ab\rangle}{\sigma_{ab}} & a \neq b \\ -\displaystyle\sum_{b\neq a} \frac{\langle ab\rangle[bq]}{\sigma_{ab}[aq]} & a = b \end{cases} \qquad (\mathfrak{C}_+)_{ab} = \begin{cases} \dfrac{[ab]}{\sigma_{ab}} & a \neq b \\ -\displaystyle\sum_{b\neq a} \frac{[ab]\langle bp\rangle}{\sigma_{ab}\langle ap\rangle} & a = b \end{cases}, \tag{6.1}$$

where p and q are two arbitrary reference spinors. Like the Hodges matrix (2.94), the choice of these reference spinors does not affect the diagonal elements as long as $\{\sigma\}$ is a solution to SE. For example, with another spinor $\tilde{q}$, we have:

$$\sum_{b\neq a} \frac{\langle ab\rangle[b\tilde{q}]}{\sigma_{ab}[a\tilde{q}]} = \sum_{b\neq a} \frac{\langle ab\rangle[b\tilde{q}][aq]}{\sigma_{ab}[a\tilde{q}][aq]} = \sum_{b\neq a} \left(\frac{\langle ab\rangle[ab][q\tilde{q}]}{\sigma_{ab}[a\tilde{q}][aq]} + \frac{\langle ab\rangle[bq]}{\sigma_{ab}[aq]} \right) = \sum_{b\neq a} \frac{\langle ab\rangle[bq]}{\sigma_{ab}[aq]}. \tag{6.2}$$

These two matrices can be understood as the generalization of the Hodges matrix. For a submatrix of $\mathfrak{C}_-$ with rows $I_r = \{i_1 \ldots i_r\}$ and columns $J_r = \{j_1 \ldots j_r\}$ deleted, its determinant scales under $|a\rangle \to t|a\rangle$ as:

$$\det[(\mathfrak{C}_-)^{i_1\ldots i_r}_{j_1\ldots j_r}] \;\to\; \det[(\mathfrak{C}_-)^{i_1\ldots i_r}_{j_1\ldots j_r}] \times \begin{cases} 1 & a \in I_r \text{ and } a \in J_r \\ t & a \in I_r \text{ or } a \in J_r \text{ (not both)} \\ t^2 & a \notin I_r \text{ and } a \notin J_r \end{cases}.$$

If $\det[(\mathfrak{C}_-)^{i_1\ldots i_r}_{j_1\ldots j_r}] \neq 0$, the rescaling will not make it vanish. On the other hand, if we have $\det[(\mathfrak{C}_-)^{i_1\ldots i_r}_{j_1\ldots j_r}] = 0$ instead, it remains so after the rescaling. Therefore, the rank of $\mathfrak{C}_\pm$ is invariant under the little group rescaling and thus depends only on the kinematics.

According to Eq. (3.31), the C matrix in the CHY integrand Ψ splits into positive and negative helicity parts as:

$$C = \begin{pmatrix} C_- \\ C_+ \end{pmatrix}, \tag{6.3}$$

at general N^{k}MHV configurations.[1] If we choose the polarization vectors as:

$$\begin{aligned} (\epsilon_a^-)_\mu &= \frac{\langle a|\overline{\sigma}_\mu|q]}{\sqrt{2}[qa]} \quad a \in \mathsf{N}^- = \{1, 2 \ldots k+2\} \\ (\epsilon_a^+)_\mu &= \frac{\langle p|\overline{\sigma}_\mu|a]}{\sqrt{2}\langle pa\rangle} \quad a \in \mathsf{N}^+ = \{k+3, k+4 \ldots n\}, \end{aligned} \tag{6.4}$$

the $(k+2) \times n$ matrix C_- and $(n-k-2) \times n$ matrix C_+ can be related to $\mathfrak{C}_\pm$ as:

$$(C_-)_{ab} = (\mathfrak{C}_-)_{ab} \frac{[bq]}{[aq]} \quad a \in \mathsf{N}^-, \qquad (C_+)_{ab} = (\mathfrak{C}_+)_{ab} \frac{\langle bp\rangle}{\langle ap\rangle} \quad a \in \mathsf{N}^+. \tag{6.5}$$

[1] Since $\mathrm{Pf}'(\Psi)$ is invariant under permutation, we can always rearrange and relabel the particles into this configuration.

In the following discussion, we will use $\mathsf{N}^{\pm}$ to denote the set of positive (negative) helicity particles. The rank of $C_{\pm}$ interplays with both the rank of $\mathfrak{C}_{\pm}$ and the helicity configurations. On one hand, the helicity configuration gives the upper bounds for rank$(C_{\pm})$:

$$\operatorname{rank}(C_{-}) \leqslant k+2 \qquad \operatorname{rank}(C_{+}) \leqslant n-k-2\,.$$

On the other hand, if $C_{\pm}$ is not of full rank, we must have:

$$\begin{aligned} \operatorname{rank}(C_{-}) &= \operatorname{rank}(\mathfrak{C}_{-}) \qquad \text{if} \;\; \operatorname{rank}(C_{-}) < k+2 \\ \operatorname{rank}(C_{+}) &= \operatorname{rank}(\mathfrak{C}_{+}) \qquad \text{if} \;\; \operatorname{rank}(C_{+}) < n-k-2\,. \end{aligned} \tag{6.6}$$

The reason is that when calculating the minors of $C_{\pm}$, the gauge dependent part can always be pulled out of the determinant. Schematically, we have:

$$\begin{aligned} \operatorname{minor}(C_{-}) &= \left(\prod_{\text{row}} \frac{1}{[aq]}\right)\left(\prod_{\text{column}} [bq]\right) \operatorname{minor}(\mathfrak{C}_{-})\,, \\ \operatorname{minor}(C_{+}) &= \left(\prod_{\text{row}} \frac{1}{\langle ap\rangle}\right)\left(\prod_{\text{column}} \langle bp\rangle\right) \operatorname{minor}(\mathfrak{C}_{+})\,. \end{aligned}$$

As a result, if the minor of $\mathfrak{C}_{\pm}$ vanishes, the corresponding minor of $C_{\pm}$ will also vanish, which leads to Eq. (6.6). Altogether, we have:

$$\operatorname{rank}(C_{-}) = \min\{k+1, \operatorname{rank}(\mathfrak{C}_{-})\} \qquad \operatorname{rank}(C_{+}) = \min\{n-k-2, \operatorname{rank}(\mathfrak{C}_{+})\}\,. \tag{6.7}$$

In general, rank(C) can be smaller that the sum of rank$(C_{\pm})$ if there exists a linear relation between rows in $C_{\pm}$. However, we can break such relations by choosing a different gauge, such that we can always make

$$\operatorname{rank}(C) = \operatorname{rank}(C_{-}) + \operatorname{rank}(C_{+}) \leqslant n-2\,, \tag{6.8}$$

since C always has two null vectors, as discussed in Section 3.3.

By simple observation, we also find that rank$(\mathfrak{C}_{\pm}) \leqslant n-2$ since they respectively have two null vectors independent of the solutions:

$$\begin{aligned} &\text{null vectors of } \mathfrak{C}_{-}: \qquad \begin{pmatrix} [1q] \\ [2q] \\ \vdots \\ [nq] \end{pmatrix} \quad \begin{pmatrix} [1q]\sigma_1 \\ [2q]\sigma_2 \\ \vdots \\ [nq]\sigma_n \end{pmatrix} \\ &\text{null vectors of } \mathfrak{C}_{+}: \qquad \begin{pmatrix} \langle 1p\rangle \\ \langle 2p\rangle \\ \vdots \\ \langle np\rangle \end{pmatrix} \quad \begin{pmatrix} \langle 1p\rangle\sigma_1 \\ \langle 2p\rangle\sigma_2 \\ \vdots \\ \langle np\rangle\sigma_n \end{pmatrix}\,. \end{aligned} \tag{6.9}$$

Actually, using Eq. (6.8), we can further prove as follows that $\operatorname{rank}(\mathfrak{C}_\pm) \leqslant n-3$ for any solution: we choose $n-1$ negative helicity particles and only one positive helicity particle, such that in C the C_- part has $n-1$ rows while the C_+ part has only one row. In this case, we must have $\operatorname{rank}(C_+) = 1$ and consequently $\operatorname{rank}(C_-) \leqslant n-3$. Since C_- is not of full rank, we must have $\operatorname{rank}(\mathfrak{C}_-) = \operatorname{rank}(C_-) \leqslant n-3$. The $\operatorname{rank}(\mathfrak{C}_+) \leqslant n-3$ can be derived in exactly the same way.

We can explicitly check that the two special solutions (4.1) and (4.2) lead to:

$$\begin{aligned} &\operatorname{rank}[\mathfrak{C}_-(\sigma^{(1)})] = 1 && \operatorname{rank}[\mathfrak{C}_-(\sigma^{(2)})] = n-3\,, \\ &\operatorname{rank}[\mathfrak{C}_+(\sigma^{(1)})] = n-3 && \operatorname{rank}[\mathfrak{C}_+(\sigma^{(2)})] = 1\,. \end{aligned} \tag{6.10}$$

Thus, the lower bound 1 and the upper bound $n-3$ of $\operatorname{rank}(\mathfrak{C}_\pm)$ can indeed be reached by some solutions.

6.2 Rank characterization

The discussion in the previous section enables us to define a partition of the solution set using $\operatorname{rank}(\mathfrak{C}_-)$:

$$\text{solution set } = \bigcup_{m=0}^{n-4} \boldsymbol{P}_-(n-3,m) \qquad \boldsymbol{P}_-(n-3,i) \cap \boldsymbol{P}_-(n-3,j) = \varnothing \ \text{ if } \ i \neq j\,, \tag{6.11}$$

such that for each $\{\sigma\} \in \boldsymbol{P}_-(n-3,m)$, we have

$$\operatorname{rank}[\mathfrak{C}_-(\sigma)] = m+1\,.$$

Similarly, $\operatorname{rank}(\mathfrak{C}_+)$ gives another partition:

$$\text{solution set } = \bigcup_{m=0}^{n-4} \boldsymbol{P}_+(n-3,m) \qquad \boldsymbol{P}_+(n-3,i) \cap \boldsymbol{P}_+(n-3,j) = \varnothing \ \text{ if } \ i \neq j\,, \tag{6.12}$$

such that for each $\{\omega\} \in \boldsymbol{P}_+(n-3,m)$, we have

$$\operatorname{rank}[\mathfrak{C}_+(\omega)] = m+1\,.$$

Intuitively, these two partitions should coincide with each other in the sense that $\boldsymbol{P}_+$ should just be a relabeling of the sets in $\boldsymbol{P}_-$. Our next task is thus to derive this identification.

Given a solution $\{\sigma\} \in \boldsymbol{P}_-(n-3,m)$, we can extract $m+1$ linearly independent rows of $\mathfrak{C}_-$ and make them into an $(m+1) \times n$ matrix C_- with $\operatorname{rank}(C_-) = m+1$. Then the C_+

part must have co-rank more then two: $\text{rank}(C_+) = \text{rank}(\mathfrak{C}_+) \leqslant n - m - 3$. Namely, the partition $\boldsymbol{P}_-$ has the following property: for each solution $\{\sigma\} \in \boldsymbol{P}_-(n-3, m)$, we must have:

$$\text{rank}[\mathfrak{C}_-(\sigma)] = m + 1 \qquad \text{rank}[\mathfrak{C}_+(\sigma)] \leqslant n - m - 3\,. \tag{6.13}$$

Similarly, for each $\{\omega\} \in \boldsymbol{P}_+(n-3, m)$, we can derive:

$$\text{rank}[\mathfrak{C}_+(\omega)] = m + 1 \qquad \text{rank}[\mathfrak{C}_-(\omega)] \leqslant n - m - 3\,. \tag{6.14}$$

Using this piece of information, we can prove that N^{k}MHV Yang-Mills and gravity amplitudes are locked with only one subset of solutions:

Theorem 6.1: *Only those solutions in the subset $\boldsymbol{P}_-(n-3, k)$ [or $\boldsymbol{P}_+(n-3, k)$] can support $Pf'(\Psi)$ at N^{k}MHV (or N^{n-k-4}MHV) configurations.*

This theorem will be proved in the next section.

Under the $3+1$ signature, we have $[\mathfrak{C}_-(\sigma)]^* = \mathfrak{C}_+(\sigma^*)$ since $(\lambda_i)_\alpha$ and $(\widetilde{\lambda})_{\dot{\alpha}}$ are complex conjugate to each other. Therefore, for each $\{\sigma\} \in \boldsymbol{P}_-(n-3, m)$, we must have:

$$\text{rank}[\mathfrak{C}_+(\sigma^*)] = m + 1\,. \tag{6.15}$$

It means that $\boldsymbol{P}^*_-(n-3, m)$, the complex conjugate of $\boldsymbol{P}_-(n-3, m)$, must be a subset of $\boldsymbol{P}_+(n-3, m)$. We can similarly derive that $\boldsymbol{P}^*_+(n-3, m)$ must be a subset of $\boldsymbol{P}_-(n-3, m)$. Therefore, we must have the relation:

$$\boldsymbol{P}^*_-(n-3, m) = \boldsymbol{P}_+(n-3, m)\,. \tag{6.16}$$

Then, according to Theorem 6.1, both $\boldsymbol{P}_-(n-3, k)$ and $\boldsymbol{P}_+(n-3, n-k-4)$ support the N^{k}MHV configuration, such that they must equal. Then using Eq. (6.16), we get:

$$\boldsymbol{P}_+(n-3, n-k-4) = \boldsymbol{P}^*_-(n-3, n-k-4) = \boldsymbol{P}_-(n-3, k)\,. \tag{6.17}$$

This leads to the identification of the two partitions:

$$\text{for all } \{\sigma\} \in \boldsymbol{P}_-(n-3, m): \qquad \text{rank}[\mathfrak{C}_-(\sigma)] = m + 1 \quad \text{rank}[\mathfrak{C}_+(\sigma)] = n - m - 3\,. \tag{6.18}$$

We note that this result can be viewed as a corollary of Theorem 6.1, and it is, of course, not used in the proof of Theorem 6.1.

6.3 Rank and Eulerian sectors

Now we need to address the last question: how many solutions does each subset $\boldsymbol{P}_-(n-3,k)$ contain? In this section, we provide a derivation[2] showing that this rank characterization can be related to the degree characterization discussed in Appendix C through

$$\begin{array}{l} \deg[\lambda_\alpha(z)] = d \\ \deg[\widetilde{\lambda}_{\dot\alpha}(z)] = n-d-2 \end{array} \iff \begin{array}{l} \mathrm{rank}(\mathfrak{C}_-) = n-d-2 \\ \mathrm{rank}(\mathfrak{C}_+) = d \end{array} , \tag{6.19}$$

with $d = n-k-3$. The number of solutions in each partition is thus

$$|\boldsymbol{P}_\pm(n-3,k)| = A(n-3,d-1) = A(n-3,k)\,. \tag{6.20}$$

We start with a generic solution $\{\sigma\}$ that makes

$$\deg[\lambda_\alpha(z)] = d \qquad \deg[\widetilde{\lambda}_{\dot\alpha}(z)] = \widetilde{d} = n-d-2\,.$$

More specifically, we have

$$\begin{aligned} (\lambda_i)_\alpha &= t_i\lambda_\alpha(\sigma_i) & \lambda_\alpha(\sigma_i) &= \sum_{l=0}^{d}(\rho_l)_\alpha\sigma_i^l\,, \\ (\widetilde{\lambda}_i)_{\dot\alpha} &= \frac{\widetilde{\lambda}_{\dot\alpha}(\sigma_i)}{t_i\prod_{j\neq i}(\sigma_i-\sigma_j)} & \widetilde{\lambda}_{\dot\alpha}(\sigma_i) &= \sum_{l=0}^{\widetilde{d}}(\widetilde{\rho}_l)_{\dot\alpha}\sigma_i^l\,, \end{aligned} \tag{6.21}$$

according to Section 2.5 and Appendix C. In the diagonal elements of $\mathfrak{C}_-$, we have

$$\begin{aligned} \frac{1}{t_j}\,[jq] &= \sum_{l=0}^{d}[\rho_l q]\sigma_j^l = [\rho_d q](\sigma_j-\sigma_{p_1})\dots(\sigma_j-\sigma_{p_d}) \\ \frac{1}{t_i}\,[iq] &= \sum_{l=0}^{d}[\rho_l q]\sigma_i^l = [\rho_d q](\sigma_i-\sigma_{p_1})\dots(\sigma_i-\sigma_{p_d})\,, \end{aligned} \tag{6.22}$$

where $\{\sigma_{p_1}\dots\sigma_{p_d}\}$ are nothing but the d zeros of the degree d polynomial $\sum_{l=0}^{d}[\rho_l q]z^l$. As a result, $(\mathfrak{C}_-)_{ii}$ becomes:

$$(\mathfrak{C}_-)_{ii} = -\sum_{j\neq i}\frac{\langle ij\rangle t_j}{\sigma_{ij}t_i}\prod_{a=1}^{d}\frac{(\sigma_j-\sigma_{p_a})}{(\sigma_i-\sigma_{p_a})}\,. \tag{6.23}$$

---

[2]This derivation is partly inspired by private communication with Freddy Cachazo.

Now the gauge freedom in $(\mathfrak{C}_-)_{ii}$ can be rephrased as the following: *the value of* $(\mathfrak{C}_-)_{ii}$ *is independent of the choice of the reference points* σ_{p_a}. To prove this, we change σ_{p_1} to $\widetilde{\sigma}_{p_1}$, and then we have:

$$\begin{aligned} & -\sum_{j\neq i} \frac{\langle ij\rangle t_j}{\sigma_{ij} t_i} \frac{\sigma_j - \widetilde{\sigma}_{p_1}}{\sigma_i - \widetilde{\sigma}_{p_1}} \prod_{a=2}^{d} \frac{(\sigma_j - \sigma_{p_a})}{(\sigma_i - \sigma_{p_a})} \\ = & -\sum_{j\neq i} \frac{\langle ij\rangle t_j}{\sigma_{ij} t_i} \frac{(\sigma_i - \sigma_{p_1})(\sigma_j - \widetilde{\sigma}_{p_1})}{(\sigma_i - \sigma_{p_1})(\sigma_i - \widetilde{\sigma}_{p_1})} \prod_{a=2}^{d} \frac{(\sigma_j - \sigma_{p_a})}{(\sigma_i - \sigma_{p_a})} \,. \end{aligned} \tag{6.24}$$

Using the Schouten identity

$$(\sigma_i - \sigma_{p_1})(\sigma_j - \widetilde{\sigma}_{p_1}) = (\sigma_i - \sigma_j)(\sigma_{p_1} - \widetilde{\sigma}_{p_1}) + (\sigma_i - \widetilde{\sigma}_{p_1})(\sigma_j - \sigma_{p_1}) \,,$$

we can write the second line of the above equation as

$$(\mathfrak{C}_-)_{ii} - \frac{\sigma_{p_1} - \widetilde{\sigma}_{p_1}}{t_i(\sigma_i - \widetilde{\sigma}_{p_1}) \prod_{a=1}^{d} (\sigma_i - \sigma_{p_a})} \sum_{j=1}^{n} \langle ij\rangle t_j \prod_{a=2}^{d} (\sigma_j - \sigma_{p_a}) \,. \tag{6.25}$$

In the summation over j, we replace $(\widetilde{\lambda}_j)_{\dot{\alpha}} t_j$ according to Eq. (6.21), such that we have:

$$\sum_{j=1}^{n} \frac{\langle i\widetilde{\lambda}(\sigma_j)\rangle \prod_{a=2}^{d} (\sigma_j - \sigma_{p_a})}{\prod_{k\neq j} (\sigma_j - \sigma_k)} \,. \tag{6.26}$$

In this numerator, $\langle i\widetilde{\lambda}(\sigma_j)\rangle$ is of degree $\widetilde{d}$, while $\prod_{a=2}^{d} (\sigma_j - \sigma_{p_a})$ is of degree $d-1$ in σ_j, such that the numerator is a degree $\widetilde{d} + d - 1 = n - 3$ polynomial of σ_j. Then, according to Eq. (2.139), this summation gives zero. Moreover, this calculation can still go through if we have one more factor $\frac{\sigma_j - \sigma_{p_0}}{\sigma_i - \sigma_{p_0}}$ in $(\mathfrak{C}_-)_{ii}$. Therefore, our conclusion is that

$$(\mathfrak{C}_-)_{ii} = -\sum_{j\neq i} \frac{\langle ij\rangle t_j}{\sigma_{ij} t_i} \prod_{a=0}^{d} \frac{(\sigma_j - \sigma_{p_a})}{(\sigma_i - \sigma_{p_a})} \tag{6.27}$$

is independent of the choice of the $d+1$ reference points $\{\sigma_{p_0} \ldots \sigma_{p_d}\}$. Eq. (6.23) can then be viewed as a special case of Eq. (6.27) with $\sigma_{p_0} \to \infty$. Now we define another matrix:

$$\widetilde{\Phi}_{ij} = (\mathfrak{C}_-)_{ij} t_i t_j \,, \tag{6.28}$$

such that

$$\widetilde{\Phi}_{ij} = \frac{\langle ij\rangle}{\sigma_{ij}} (t_i t_j) \quad (i \neq j) \qquad\qquad \widetilde{\Phi}_{ii} = -\sum_{j\neq i} \frac{\langle ij\rangle}{\sigma_{ij}} (t_i t_j) \prod_{a=0}^{d} \frac{(\sigma_j - \sigma_{p_a})}{(\sigma_i - \sigma_{p_a})} \,. \tag{6.29}$$

This matrix agrees with the $\widetilde{\Phi}_{ij}$ defined in [27, 28], and it has the same rank as $\mathfrak{C}_-$.

Next, we come to prove our main result of this section, Eq. (6.19). First, to prove that

$$\text{rank}(\mathfrak{C}_-) = \text{rank}(\widetilde{\Phi}) = n - d - 2\,,$$

we only need to find the $d+2$ linearly independent vectors that are annihilated by $\widetilde{\Phi}$. In hindsight, we claim that these vectors are:

$$v_j(m) = \sigma_j^m \qquad (0 \leqslant m \leqslant d+1)\,, \tag{6.30}$$

and show that indeed

$$\sum_{j=1}^{n} \widetilde{\Phi}_{ij} v_j(m) = 0\,. \tag{6.31}$$

First, we have

$$\begin{aligned}\sum_{j=1}^{n} \widetilde{\Phi}_{ij} v_j &= \sum_{j\neq i} \widetilde{\Phi}_{ij} v_j + \widetilde{\Phi}_{ii} v_i \\ &= \sum_{j\neq i} \frac{\langle ij\rangle t_i t_j \sigma_j^m}{\sigma_{ij}} - \sum_{j\neq i} \frac{\langle ij\rangle t_i t_j \sigma_i^m}{\sigma_{ij}} \prod_{a=0}^{d} \frac{(\sigma_j - \sigma_{p_a})}{(\sigma_i - \sigma_{p_a})}\,.\end{aligned} \tag{6.32}$$

In the first term of Eq. (6.32), we substract and add σ_i^m to the numerator, such that

$$\begin{aligned}\sum_{j\neq i} \frac{\langle ij\rangle t_i t_j \sigma_j^m}{\sigma_{ij}} &= \sum_{j\neq i} \frac{\langle ij\rangle t_i t_j (\sigma_j^m - \sigma_i^m)}{\sigma_{ij}} + \sum_{j\neq i} \frac{\langle ij\rangle t_i t_j \sigma_i^m}{\sigma_{ij}} \\ &= t_i \sum_{j\neq i} \frac{\langle i\widetilde{\lambda}(\sigma_j)\rangle (\sigma_j^m - \sigma_i^m)}{\sigma_{ij}\prod_{k\neq j}(\sigma_j - \sigma_k)} + \sum_{j\neq i} \frac{\langle ij\rangle t_i t_j \sigma_i^m}{\sigma_{ij}}\,.\end{aligned} \tag{6.33}$$

Now in the first term of Eq. (6.33), since

$$\lim_{\sigma_j\to\sigma_i} \langle i\widetilde{\lambda}(\sigma_j)\rangle = \langle i\widetilde{\lambda}(\sigma_i)\rangle = \langle ii\rangle t_i \prod_{j\neq i} \sigma_{ij} = 0\,,$$

we can factorize out a factor of $(\sigma_i - \sigma_j)$ from it, namely

$$\langle i\widetilde{\lambda}(\sigma_j)\rangle = (\sigma_i - \sigma_j)\langle i\widetilde{\lambda}_\star(\sigma_j)\rangle \qquad \det[\widetilde{\lambda}_\star(\sigma_j)] = n - d - 3\,. \tag{6.34}$$

Therefore, the summation in gives

$$\sum_{j\neq i} \frac{\langle i\widetilde{\lambda}(\sigma_j)\rangle (\sigma_j^m - \sigma_i^m)}{\sigma_{ij}\prod_{k\neq j}(\sigma_j - \sigma_k)} = \sum_{j=1}^{n} \frac{\langle i\widetilde{\lambda}_\star(\sigma_j)\rangle (\sigma_j^m - \sigma_i^m)}{\prod_{k\neq j}(\sigma_j - \sigma_k)}\,, \tag{6.35}$$

which is zero according to Eq. (2.139) since the numerator always has degree no more than $n-2$ in σ_j. In addition, we can restore $j=i$ in the summation because this numerator vanishes at $j=i$. Therefore, Eq. (6.33) leads to

$$\begin{aligned}\sum_{j\neq i}\frac{\langle ij\rangle t_i t_j \sigma_j^m}{\sigma_{ij}} = \sum_{j\neq i}\frac{\langle ij\rangle t_i t_j \sigma_i^m}{\sigma_{ij}} &= t_i\sigma_i^m\sum_{j\neq i}\frac{\langle i\widetilde{\lambda}(\sigma_j)\rangle}{\sigma_{ij}\prod_{k\neq j}(\sigma_j-\sigma_k)}\\ &= t_i\sigma_i^m\sum_{j\neq i}\frac{\langle i\widetilde{\lambda}_\star(\sigma_j)\rangle}{\prod_{k\neq j}(\sigma_j-\sigma_k)}\\ &= -t_i\sigma_i^m\frac{\langle i\widetilde{\lambda}_\star(\sigma_i)\rangle}{\prod_{k\neq i}(\sigma_i-\sigma_k)}\,,\end{aligned} \tag{6.36}$$

where we have used again Eq. (2.139) in the last line. This gives the final form of the first term in Eq. (6.32). The second term in Eq. (6.32) can be transformed as

$$\begin{aligned}\sum_{j\neq i}\frac{\langle ij\rangle t_i t_j \sigma_i^m}{\sigma_{ij}}\prod_{a=0}^{d}\frac{(\sigma_j-\sigma_{p_a})}{(\sigma_i-\sigma_{p_a})} &= \frac{t_i\sigma_i^m}{\prod_a(\sigma_i-\sigma_{p_a})}\sum_{j\neq i}\frac{\langle i\widetilde{\lambda}(\sigma_j)\rangle\prod_a(\sigma_j-\sigma_{p_a})}{\sigma_{ij}\prod_{k\neq j}(\sigma_j-\sigma_k)}\\ &= \frac{t_i\sigma_i^m}{\prod_a(\sigma_i-\sigma_{p_a})}\sum_{j\neq i}\frac{\langle i\widetilde{\lambda}_\star(\sigma_j)\rangle\prod_a(\sigma_j-\sigma_{p_a})}{\prod_{k\neq j}(\sigma_j-\sigma_k)}\\ &= -t_i\sigma_i^m\frac{\langle i\widetilde{\lambda}_\star(\sigma_i)\rangle}{\prod_{k\neq i}(\sigma_i-\sigma_k)}\,.\end{aligned} \tag{6.37}$$

Therefore, the first and second term in (6.32) equal to each other, such that we have proved that

$$\sum_{j=1}^{n}\widetilde{\Phi}_{ij}v_j(m)=0 \qquad (0\leqslant m\leqslant d+1)\,. \tag{6.38}$$

Now we have shown that $\widetilde{\Phi}$ has $d+2$ null vectors $v_j(m)$, which leads to

$$\begin{matrix}\deg[\lambda_\alpha(z)]=d\\ \deg[\widetilde{\lambda}_{\dot\alpha}(z)]=n-d-2\end{matrix} \quad\Longrightarrow\quad \mathrm{rank}(\widetilde{\Phi})=\mathrm{rank}(\mathfrak{C}_-)=n-d-2\,. \tag{6.39}$$

The opposite direction also holds since every step can be reversed.

For $\mathfrak{C}_+$, we can similarly define the matrix

$$\Phi_{ij}=(\mathfrak{C}_+)_{ij}(t_it_j)^{-1} \tag{6.40}$$

such that

$$\Phi_{ij}=\frac{[ij]}{\sigma_{ij}t_it_j}\quad (i\neq j) \qquad \Phi_{ii}=-\sum_{j\neq i}\frac{[ij]}{\sigma_{ij}t_it_j}\frac{\prod_{k\neq i}(\sigma_i-\sigma_k)}{\prod_{k\neq j}(\sigma_j-\sigma_k)}\prod_{a=0}^{\tilde d}\frac{(\sigma_j-\sigma_{p_a})}{(\sigma_i-\sigma_{p_a})}\,. \tag{6.41}$$

This matrix agrees with the Φ in [27,28]. It has the following independent $n-d$ null vectors:

$$v_j(m) = \frac{\sigma_j^m}{\prod_{k\neq j}(\sigma_j - \sigma_k)} \qquad (0 \leqslant m \leqslant n-d-1)\,, \tag{6.42}$$

such that

$$\mathrm{rank}(\Phi) = \mathrm{rank}(\mathfrak{C}_+) = d\,. \tag{6.43}$$

This completes our proof of Eq. (6.19).

6.4 Relating solution sectors to helicity sectors

In this section, we give a proof to Theorem 6.1. In the first part of the proof, we consider a solution $\{\sigma\} \in \boldsymbol{P}_-(n-3,m)$ with $m \leqslant k$, which gives C_- nonzero co-rank. We will show that $\mathrm{Pf}'(\Psi) \neq 0$ only when $m = k$. For convenience, we use $r \equiv m+1$ to denote the rank.

Under the gauge choice Eq. (6.4), the matrix Ψ has the following form at N^{k}MHV:

$$\Psi = \begin{pmatrix} A & -C_-^T & -C_+^T \\ C_- & 0 & \mathcal{B} \\ C_+ & -\mathcal{B}^T & 0 \end{pmatrix}\,. \tag{6.44}$$

While $C_\pm$ has already been given in Eq. (6.5), the $n \times n$ matrix A is given by:

$$A_{ab} = -\frac{\langle ab\rangle [ab]}{\sigma_{ab}} \qquad a,b \in \mathsf{N}\,, \tag{6.45}$$

and the $(k+2)\times(n-k-2)$ matrix $\mathcal{B}$ is given by:

$$\mathcal{B}_{ab} = -\frac{\langle ap\rangle[bq]}{[aq]\langle bp\rangle \sigma_{ab}} \qquad a \in \mathsf{N}^- \quad b \in \mathsf{N}^+\,. \tag{6.46}$$

Since $\mathrm{rank}(C_-) = r$, we can choose in the C_- part an $r \times r$ reference matrix:

$$\mathcal{R} = \begin{pmatrix} (C_-)_{i_1 j_1} & (C_-)_{i_1 j_2} & \cdots & (C_-)_{i_1 j_r} \\ \vdots & \vdots & & \vdots \\ (C_-)_{i_r j_1} & (C_-)_{i_r j_2} & \cdots & (C_-)_{i_r j_r} \end{pmatrix}\,, \tag{6.47}$$

where the row and column indices are in the set:

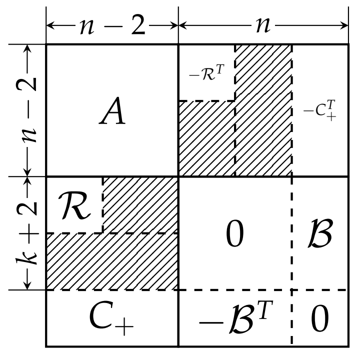
$$\mathsf{I}_r = \{i_1, i_2 \ldots i_r\} \qquad \mathsf{J}_r = \{j_1, j_2 \ldots j_r\}\,.$$

The determinant of $\mathcal{R}$ can be related to the corresponding minor of $\mathfrak{C}_-$ through:

$$\det(\mathcal{R}) = \left(\prod_{k=1}^{r} \frac{[j_k q]}{[i_k q]}\right) \det[(\mathfrak{C}_-)_{\mathsf{I}_r \mathsf{J}_r}]\,, \quad (\mathfrak{C}_-)_{\mathsf{I}_r \mathsf{J}_r} \equiv \begin{pmatrix} (\mathfrak{C}_-)_{i_1 j_1} & (\mathfrak{C}_-)_{i_1 j_2} & \cdots & (\mathfrak{C}_-)_{i_1 j_r} \\ \vdots & \vdots & & \vdots \\ (\mathfrak{C}_-)_{i_r j_1} & (\mathfrak{C}_-)_{i_r j_2} & \cdots & (\mathfrak{C}_-)_{i_r j_r} \end{pmatrix}\,. \tag{6.48}$$

Next, we delete the $(n-1)$-th and n-th row and column in Ψ. Before any further manipulation, this matrix looks like:

$$\boldsymbol{\psi} \equiv \Psi^{n-1,n}_{n-1,n} = \qquad (6.49)$$



where we have already moved $\mathcal{R}$ to the upper left corner of C_- through permutations. Since $\mathcal{R}$ has full rank, we can use it to eliminate all the elements in the gray shaded region in Eq. (6.49) by the following two elementary transformations:

$$\boldsymbol{\psi} \;\to\; (\boldsymbol{P}_2^T\boldsymbol{P}_1)\boldsymbol{\psi}(\boldsymbol{P}_1^T\boldsymbol{P}_2)\,. \qquad (6.50)$$

The $(2n-2)\times(2n-2)$ matrix $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ have the following form:

$$\boldsymbol{P}_1 = \begin{pmatrix} \mathbf{1}_{(n-2)} & & & \\ & \mathbf{1}_r & 0 & \\ & -\boldsymbol{y} & \mathbf{1}_{(k+2-r)} & \\ & & & \mathbf{1}_{(n-k-2)} \end{pmatrix} \qquad \boldsymbol{P}_2 = \begin{pmatrix} \mathbf{1}_r & -\boldsymbol{x} & \\ 0 & \mathbf{1}_{n-2-r} & \\ & & \mathbf{1}_n \end{pmatrix}, \qquad (6.51)$$

where $\mathbf{1}_{(n-2)}$ is the $(n-2)\times(n-2)$ unit matrix, for example. The matrix $\boldsymbol{x}$ and $\boldsymbol{y}$ are solved from the following two linear equations:

$$\mathcal{R}\boldsymbol{x} = \begin{pmatrix} (C_-)_{i_1,\underline{r+1}} & \cdots & (C_-)_{i_1,\underline{n-2}} \\ \vdots & & \vdots \\ (C_-)_{i_r,\underline{r+1}} & \cdots & (C_-)_{i_r,\underline{n-2}} \end{pmatrix}, \qquad (6.52)$$

$$\boldsymbol{y}\mathcal{R} = \begin{pmatrix} (C_-)_{\overline{r+1},j_1} & \cdots & (C_-)_{\overline{r+1},j_r} \\ \vdots & & \vdots \\ (C_-)_{\overline{k+2},j_1} & \cdots & (C_-)_{\overline{k+2},j_r} \end{pmatrix}, \qquad (6.53)$$

where the underlined and overlined indices are taken from the range:

$$\{\underline{r+1},\ldots,\underline{n-2}\} = \{1,2\ldots n-2\}\setminus \mathsf{J}_r\,, \qquad \{\overline{r+1},\ldots,\overline{k+2}\} = \{1,2\ldots k+2\}\setminus \mathsf{I}_r\,.$$

The expression for $\boldsymbol{y} = (y_{ij})$ will not be used in the following, while the expression for $\boldsymbol{x}$ can be derived from the Cramer's rule:

$$x_{i_k a} = \frac{1}{\det(\mathfrak{C}^r_-)}\frac{[aq]}{[j_k q]}\det\begin{bmatrix} j_k \\ \boldsymbol{c}_a \end{bmatrix}, \qquad i_k \in \mathsf{I}_r, \quad a \in \{1,2,\ldots,n-2\}\setminus \mathsf{J}_r\,. \qquad (6.54)$$

Here and later in this chapter, we use the notation:

$$\begin{bmatrix} j_1 & j_2 & \cdots \\ \boldsymbol{c}_{a_1} & \boldsymbol{c}_{a_2} & \cdots \end{bmatrix} \tag{6.55}$$

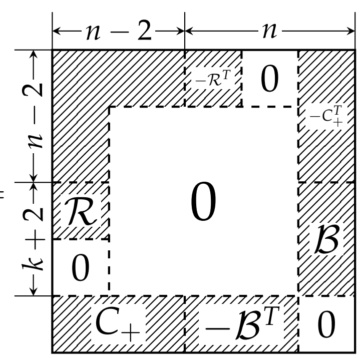
to stand for the matrix obtained from $(\mathfrak{C}_-)_{\mathsf{I}_r\mathsf{J}_r}$ with the columns

$$\boldsymbol{c}_{j_1} \equiv \begin{pmatrix} (\mathfrak{C}_-)_{i_1 j_1} \\ (\mathfrak{C}_-)_{i_2 j_1} \\ \vdots \\ (\mathfrak{C}_-)_{i_r j_1} \end{pmatrix} \qquad \boldsymbol{c}_{j_2} \equiv \begin{pmatrix} (\mathfrak{C}_-)_{i_1 j_2} \\ (\mathfrak{C}_-)_{i_2 j_2} \\ \vdots \\ (\mathfrak{C}_-)_{i_r j_2} \end{pmatrix} \qquad \cdots$$

by the columns $\{\boldsymbol{c}_{a_1}, \boldsymbol{c}_{a_2} \ldots\}$. After the transformation (6.50), the gray shaded region becomes zero as designed. However, the lower right corner of A part also becomes zero, due to the identity:

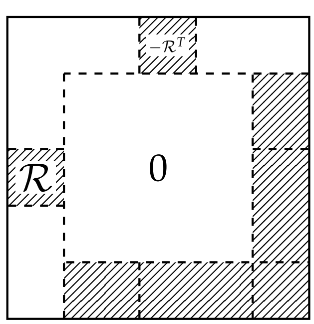
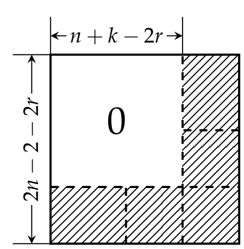
$$A_{ab} - \sum_{k=1}^{r} \left(A_{a j_k} x_{i_k b} + A_{j_k b} x_{i_k a} \right) + \sum_{k,s=1}^{r} x_{i_k a} A_{j_k j_s} x_{i_s b} = 0 \qquad a, b \in \{1, 2 \ldots n-2\} \backslash \mathsf{J}_r \,. \tag{6.56}$$

The detailed proof of this identity can be found in our work [41]. Therefore, after the elementary transformation (6.50), the shape of $\boldsymbol{\psi}$ becomes:

$\boldsymbol{\psi} =$  , (6.57)

where only the shaded regions are in general nonzero.

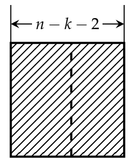
Finally, after using $\mathcal{R}$ to make the first r rows and columns entirely zero except for $\mathcal{R}$ and $-\mathcal{R}^T$ themselves, we can factorize the Pfaffian into two parts:

$\mathrm{Pf}\,(\boldsymbol{\psi}) = \mathrm{Pf}$  $= \det(\mathcal{R})\,\mathrm{Pf}$  , (6.58)

(up to a minus sign)

where the dashed lines separate the original B and C parts. However, the elements in these regions in general have very complicated expressions. In this result, the nonzero lower left block has the dimension $(n-k-2)\times(n+k-2r)$, which has more columns than rows when $r \leqslant k$. Therefore, we must have $\mathrm{Pf}'(\Psi) \propto \mathrm{Pf}\,(\boldsymbol{\psi}) = 0$ for $r \leqslant k$. Only when $r = k+1$, we can possibly have a nonzero result:

$$\mathrm{Pf}\,(\boldsymbol{\psi}) = \det(\mathcal{R})\det$$



$$\text{(up to a minus sign).} \tag{6.59}$$

Up to now, we have proved that if $m \leqslant k$, we have $\mathrm{Pf}'(\Psi) \neq 0$ only with $m = k$.

As the second part of the proof, we study the case with $m > k$. Now the C_- part always has the full rank, while the C_+ part at least has co-rank 2:

$$\mathrm{rank}(C_+) = r' \leqslant n-k-4\,.$$

After permuting the C_+ with C_- part, we can perform exactly the same operation as the first part on C_-, with angular and square brackets exchange. The final result still has the shape of Eq. (6.58), but with different dimensions:

$$\begin{aligned} &\text{upper left zero block :} && (2n-k-2r'-4)\times(2n-k-2r'-4) \\ &\text{lower left block :} && (k+2)\times(2n-k-2r'-4)\,. \end{aligned}$$

With $r' \leqslant n-k-4$, the lower left block always have more columns than rows such that $\mathrm{Pf}'(\Psi) \propto \mathrm{Pf}\,(\boldsymbol{\psi}) = 0$ for all $m > k$.

Therefore, we have completed the proof that $\mathrm{Pf}'(\Psi) \neq 0$ only with $m = k$ at N^kMHV. The relation between $\boldsymbol{P}_+$ and N^{n-k-4}MHV can also be prove identically, which completes the proof for Theorem 6.1.

6.5 General single trace Einstein-Yang-Mills amplitudes

In the single trance EYM integrand, $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ shares a very similar structure to $\mathrm{Pf}'(\Psi)$. Using the technique we have just described, we can also study the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ on the solution set at different graviton helicity configurations.

According to the graviton helicities, we can divide the matrix C_{H} in Ψ_{H} into two parts:

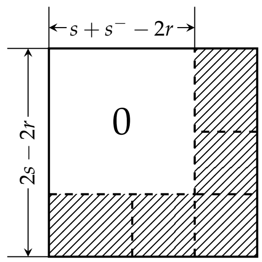
$$C_{\mathsf{H}} = \begin{pmatrix} (C_{\mathsf{H}})_- \\ (C_{\mathsf{H}})_+ \end{pmatrix} \qquad \begin{array}{ll} (C_{\mathsf{H}})_- : & s^- \times s \text{ matrix} \\ (C_{\mathsf{H}})_+ : & s^+ \times s \text{ matrix} \end{array}\,, \tag{6.60}$$

where $s^{\pm}$ is the number of positive (negative) helicity gravitons. For a solution $\{\sigma\}$ in the subset $\boldsymbol{P}_{-}(n-3,m)$, the ranks of $(C_{\mathsf{H}})_{\pm}$ must satisfy:

$$\begin{aligned} r &= \operatorname{rank}[(C_{\mathsf{H}})_{-}] = \min\{m+1,\, s^{-}\} \\ r' &= \operatorname{rank}[(C_{\mathsf{H}})_{+}] = \min\{n-m-3,\, s^{+}\}\,. \end{aligned} \tag{6.61}$$

Note that we have used Eq. (6.18), the corollary of Theorem 6.1. Then we choose an $r \times r$ reference matrix $\mathcal{R}_{\mathsf{H}}$ in the $(C_{\mathsf{H}})_{-}$ part, and perform the same transformation as in Section 6.4. The result is:

$$\mathrm{Pf}\,(\Psi_{\mathsf{h}}) = \det(\mathcal{R}_{\mathsf{h}})\,\mathrm{Pf}$$



$$(\text{up to a minus sign})\,. \tag{6.62}$$

The lower left block has the dimension $s^{+} \times s + s^{-} - 2r$ such that there are more columns than rows if $r \leqslant s^{-} - 1$. It means that $\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = 0$ if the $(C_{\mathsf{H}})_{-}$ part has a nonzero co-rank. We can also perform the same transformation onto the $(C_{\mathsf{H}})_{+}$ part, and the result is that $\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = 0$ also when the $(C_{\mathsf{H}})_{+}$ part has a nonzero co-rank: $r' \leqslant s^{+} - 1$. Therefore, we have $\mathrm{Pf}\,(\Psi_{\mathsf{H}}) = 0$ if $m \leqslant s^{-} - 2$ or $m \geqslant n - s^{+} - 2$. The support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ is thus:

$$\mathrm{Pf}\,(\Psi_{\mathsf{H}}) \neq 0 \;\text{ on the solution set } \bigcup_{m=s^{-}-1}^{n-3-s^{+}} \boldsymbol{P}_{-}(n-3,m)\,. \tag{6.63}$$

In Table 6.1, we show this pattern for $n = 7$ and $n = 8$ with all possible graviton helicity configurations. In this table, bold face numbers denote the order of the subsets. The barred subsets are the complex conjugate of the unbarred ones.

Knowing the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$, we can very easily prove that if gluons have the same helicity, the single trace EYM amplitudes must vanish. This is actually true for all N^{k}MHV configurations. Suppose all gluons have positive helicity at N^{k}MHV, then we must have $s^{-} = k + 2$ such that the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ becomes:

$$\mathrm{Pf}\,(\Psi_{\mathsf{H}}) \neq 0 \;\text{ on the solution set } \bigcup_{m=k+1}^{n-3-s^{+}} \boldsymbol{P}_{-}(n-3,m) \;\text{ if gluons are all-plus.} \tag{6.64}$$

Since the set $\boldsymbol{P}_{-}(n-3,k)$, the support of $\mathrm{Pf}'(\Psi)$ at N^{k}MHV, is not in the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$, we must have $\mathrm{Pf}\,(\Psi_{\mathsf{H}})\,\mathrm{Pf}'(\Psi) = 0$ such that the amplitude must vanish. Similarly, if gluons

Table 6.1. The solution sets that support $\mathrm{Pf}\,(\Psi_{\mathsf{h}})$ at different graviton helicity configurations at $n=7$ and $n=8$. The s^+ and s^- are the numbers of positive and negative helicity gravitons. The amplitude vanishes if the graviton number is more than $n-2$.

	$n=7$, $\mathrm{Pf}\,(\Psi_{\mathsf{h}})\neq 0$					
	$s^-=0$	$s^-=1$	$s^-=2$	$s^-=3$	$s^-=4$	$s^-=5$
$s^+=0$	$\mathbf{24}$	$\mathbf{24}$	$\mathbf{11}+\overline{\mathbf{11}}+\bar{\mathbf{1}}$	$\overline{\mathbf{11}}+\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\varnothing$
$s^+=1$	$\mathbf{24}$	$\mathbf{24}$	$\mathbf{11}+\overline{\mathbf{11}}+\bar{\mathbf{1}}$	$\overline{\mathbf{11}}+\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	
$s^+=2$	$\mathbf{1}+\mathbf{11}+\overline{\mathbf{11}}$	$\mathbf{1}+\mathbf{11}+\overline{\mathbf{11}}$	$\mathbf{11}+\overline{\mathbf{11}}$	$\overline{\mathbf{11}}$		
$s^+=3$	$\mathbf{1}+\mathbf{11}$	$\mathbf{1}+\mathbf{11}$	$\mathbf{11}$			
$s^+=4$	$\mathbf{1}$	$\mathbf{1}$				
$s^+=5$	$\varnothing$					

	$n=8$, $\mathrm{Pf}\,(\Psi_{\mathsf{h}})\neq 0$						
	$s^-=0$	$s^-=1$	$s^-=2$	$s^-=3$	$s^-=4$	$s^-=5$	$s^-=6$
$s^+=0$	$\mathbf{120}$	$\mathbf{120}$	$\mathbf{26}+\mathbf{66}+\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\mathbf{66}+\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\varnothing$
$s^+=1$	$\mathbf{120}$	$\mathbf{120}$	$\mathbf{26}+\mathbf{66}+\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\mathbf{66}+\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\overline{\mathbf{26}}+\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	
$s^+=2$	$\mathbf{1}+\mathbf{26}+\mathbf{66}+\overline{\mathbf{26}}$	$\mathbf{1}+\mathbf{26}+\mathbf{66}+\overline{\mathbf{26}}$	$\mathbf{26}+\mathbf{66}+\overline{\mathbf{26}}$	$\mathbf{66}+\overline{\mathbf{26}}$	$\overline{\mathbf{26}}$		
$s^+=3$	$\mathbf{1}+\mathbf{26}+\mathbf{66}$	$\mathbf{1}+\mathbf{26}+\mathbf{66}$	$\mathbf{26}+\mathbf{66}$	$\mathbf{66}$			
$s^+=4$	$\mathbf{1}+\mathbf{26}$	$\mathbf{1}+\mathbf{26}$	$\mathbf{26}$				
$s^+=5$	$\mathbf{1}$	$\mathbf{1}$					
$s^+=6$	$\varnothing$						

are all-minus, then the number of positive helicity gravitons must be $s^+=n-k-2$ such that the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ becomes:

$$\mathrm{Pf}\,(\Psi_{\mathsf{H}})\neq 0 \;\text{ on the solution set } \bigcup_{m=s^--1}^{k-1}\boldsymbol{P}_{-}(n-3,m) \;\text{ if gluons are all-minus.} \tag{6.65}$$

Again, the set $\boldsymbol{P}_{-}(n-3,k)$ is not in the support of $\mathrm{Pf}\,(\Psi_{\mathsf{H}})$ and this amplitude must also vanish. Therefore, we have completed the proof that *the single trace tree amplitudes of EYM with gluons have the same helicity must vanish.* This result is first argued in [49] using the factorization properties at soft and collinear limit. Using the BCFW recursive relation [44], we should also be able to give an inductive proof.

CHAPTER 7

DISCUSSION AND CONCLUSION

We first mention some possible future directions in the CHY formalism. For string theorists, a question they may ask is what the UV completion of CHY is. The textbook bosonic and fermionic strings do not reproduce the CHY formula for amplitudes. In four dimensions, the UV completion may be the topological string as proposed in Witten-RSV. However, CHY is written in a form that is valid in any dimensions, while Witten-RSV, using explicitly the spinor helicity variables, only works in four dimensions.[1] The ambitwistor string theory [68, 70], which can be defined in any dimension, seems to be a promising candidate. The field theory limit in the ambitwistor string comes at $\alpha' \to \infty$, opposite to the usual $\alpha' \to 0$ limit. One may ask whether there exists any kind of duality between these string theories. Moreover, at strong gauge coupling, we have the famous gauge/gravity, or AdS/CFT duality [88]. At weak gauge coupling, we have instead double copy relations, as we have intensively explored in the thesis. It is then interesting to ask how these relations interplay, or even get integrated into one unified treatment.

Another interesting direction is to study the properties of the solutions to SE. Although the SE has a very compact form, it is actually very hard to solve. Except for the two rational solutions (4.1) and (4.2) in four dimensions, the analytical expressions for the other solutions are not available. On the other hand, solving the SE is equivalent to finding the equilibrium positions of a system of two dimensional hard core gas. At this preliminary stage, one can ask whether we can make all the punctures $\{\sigma\}$ real in one solution. Along this line of thought, Cachazo's group proposed that the positive kinematics can make all the $(n-3)!$ solutions real [89]. However, the positive kinematics cannot be realized in the Minkowskian signature, and thus it is not physical. We find that some of the solutions can still be real even inside the physical kinematic space, and it seems that the number of

[1] The attempt to prove the equivalence in four dimensions can be found in [64, 87].

real solutions can be used to characterize certain region of the kinematic space bounded by multilinear configurations. This investigation is still ongoing.

As to the big picture, we hope the study of scattering amplitude can provide us some hints on the nature of quantum field theory. The best hope is to finally reformulate the field theory in a form that respect the full mathematical structure and symmetry of scattering amplitude. There are some success in our spherical cow: $\mathcal{N} = 4$ super Yang-Mills, while a lot of work still need to be done in more general and less symmetric field theories. The CHY formalism will be very useful since it can cope with a large class of theories beyond the gauge and gravity.

To conclude, we have shown that the CHY style direct evaluation can lead to the correct MHV gauge and gravity amplitudes. Each component involved in the CHY integrand can factorize into a gauge independent physical part and an $SL(2,\mathbb{C})$ world sheet gauge dependent part. The $SL(2,\mathbb{C})$ dependence cancels explicitly. Using the same technique, we derived a new formula for the single trace MHV amplitudes for EYM, which is equivalent to, but much simpler than, the SBDW prescription in the literature. Finally, we studied why in four dimensions, solutions can be categorized by the helicity configuration. The link is made explicit by the rank of our discriminant matrices.

APPENDIX A

PERMUTATIONS IN CHY

This appendix is devoted to verify that the reduced determinant $\det'(\Phi)$ defined in Eq. (3.27) and the reduced Pfaffian $\mathrm{Pf}'(\Psi)$ defined in Eq. (3.32) are both invariant under the choice of which rows and column being deleted. As a result, they are both invariant under permutations.

We start with the $\det'(\Phi)$:

Proposition A.1: *The reduced determinant* $\det'(\Phi)$ *is independent of the choice on which three rows and columns being deleted.*

According to the way $\det'(\Phi)$ being derived in Section 3.2, it should not depend on such a choice. However, it is instructive to check it afterwards.

proof of Proposition A.1. Since Φ is symmetric, it is sufficient to fix the deleted columns $\{i, j, k\}$ and show that

$$\frac{\det\left(\Phi^{ijk}_{pqr}\right)}{\sigma_{ij}\sigma_{jk}\sigma_{ki}} = -\frac{\det\left(\Phi^{ij,k+1}_{pqr}\right)}{\sigma_{ij}\sigma_{j,k+1}\sigma_{k+1,i}} \qquad (i < j < k)\,, \tag{A.1}$$

since the generic case is generated by succeesive neighboring permutations. Next, we multiply the $(k-2)$-th row of Φ^{ijk}_{pqr} by $\sigma_{j,k+1}\sigma_{k+1,i}$ and define it as Φ'. Similarly, we multiply the $(k-2)$-th row of $\Phi^{ij,k+1}_{pqr}$ by $\sigma_{jk}\sigma_{ki}$ and define it as Φ''. The determinant of these two new matrices satisfy:

$$\det(\Phi') = \sigma_{j,k+1}\sigma_{k+1,i}\det\left(\Phi^{ijk}_{pqr}\right) \qquad \det(\Phi'') = \sigma_{jk}\sigma_{ki}\det\left(\Phi^{ij,k+1}_{pqr}\right).$$

The matrix Φ' and Φ'' differ only by their $(k-2)$-th row:

$$\begin{aligned}
[\Phi']_{k-2}: &\qquad \sigma_{j,k+1}\sigma_{k+1,i} \times \begin{pmatrix} \frac{s_{k+1,1}}{\sigma^2_{k+1,1}} & \frac{s_{k+1,2}}{\sigma^2_{k+1,2}} & \frac{s_{k+1,3}}{\sigma^2_{k+1,3}} & \cdots & \frac{s_{k+1,n}}{\sigma^2_{k+1,n}} \end{pmatrix} \\
[\Phi'']_{k-2}: &\qquad \sigma_{jk}\sigma_{ki} \times \begin{pmatrix} \frac{s_{k1}}{\sigma^2_{k1}} & \frac{s_{k2}}{\sigma^2_{k2}} & \frac{s_{k3}}{\sigma^2_{k3}} & \cdots & \frac{s_{kn}}{\sigma^2_{kn}} \end{pmatrix}.
\end{aligned}$$

In Φ', we multiply the row starting with s_{m1}/σ^2_{m1} by $\sigma_{jm}\sigma_{mi}$ and add it up to the $(k-2)$-th row. This operation does not change the determinant while the $(k-2)$-th row becomes:

- The columns of s_{mi}/σ_{mi}^2 and s_{mj}/σ_{mj}^2 do not contain the original diagonal elements, such that the result is:

$$\sum_{m\neq i,j,k} \frac{s_{mi}}{\sigma_{mi}^2}\sigma_{jm}\sigma_{mi} = \sum_{m\neq i,j,k}\left(\sigma_{ji}\frac{s_{mi}}{\sigma_{mi}} - s_{mi}\right) = \sigma_{ij}\frac{s_{ki}}{\sigma_{ki}} + s_{ki} = -\sigma_{jk}\sigma_{ki}\frac{s_{ki}}{\sigma_{ki}^2}$$
$$\sum_{m\neq i,j,k} \frac{s_{mj}}{\sigma_{mj}^2}\sigma_{jm}\sigma_{mi} = \sum_{m\neq i,j,k}\left(\sigma_{ij}\frac{s_{mj}}{\sigma_{mj}} - s_{mj}\right) = \sigma_{ji}\frac{s_{kj}}{\sigma_{kj}} + s_{kj} = -\sigma_{jk}\sigma_{ki}\frac{s_{kj}}{\sigma_{kj}^2}\,.$$

- The column of s_{mk}/σ_{mk}^2 does not contain the origianl diagonal elements either, but it turns out to be:

$$\begin{aligned}\sum_{m\neq i,j,k}\sigma_{jm}\sigma_{mi}\frac{s_{mk}}{\sigma_{mk}^2} &= \sigma_{jk}\sigma_{ki}\sum_{m\neq i,j,k}\frac{s_{mk}}{\sigma_{mk}^2} + \sum_{m\neq i,j,k}\left(\sigma_{jk}\frac{s_{mk}}{\sigma_{mk}} + \sigma_{ik}\frac{s_{mk}}{\sigma_{mk}} - s_{mk}\right)\\ &= \sigma_{jk}\sigma_{ki}\sum_{m\neq k}\frac{s_{km}}{\sigma_{km}^2}\,.\end{aligned}$$

- For the other columns with label l, there must be one original diagonal element Φ_{ll} in the summation, such that the result is:

$$-\sigma_{jl}\sigma_{li}\sum_{m\neq l}\frac{s_{lm}}{\sigma_{lm}^2} + \sum_{m\neq k,l}\sigma_{jm}\sigma_{mi}\frac{s_{ml}}{\sigma_{ml}^2} = -\sigma_{jk}\sigma_{ki}\frac{s_{kl}}{\sigma_{kl}^2} + \sum_{m\neq l}\left(\sigma_{jm}\sigma_{mi} - \sigma_{jl}\sigma_{li}\right)\frac{s_{ml}}{\sigma_{ml}^2}$$

  The second term yiels zero since $\sigma_{jm}\sigma_{mi} = (\sigma_{jl} - \sigma_{ml})(\sigma_{ml} + \sigma_{li})$ and

$$\sum_{m\neq l}\left(\sigma_{jm}\sigma_{mi} - \sigma_{jl}\sigma_{li}\right)\frac{s_{ml}}{\sigma_{ml}^2} = \sum_{m\neq l}\left[(\sigma_{jl} - \sigma_{li})\frac{s_{ml}}{\sigma_{ml}} - s_{ml}\right] = 0$$

After this operation, we find that the $(k-2)$-th row of Φ' becomes the negative of that of Φ'', while all the other elements are identical. Since this elementary transformation does not change the determinant, we have just proved that:

$$\det(\Phi') = -\det(\Phi'')\,.$$

Eq. (A.1) thus follows immediately such that we have proved that $\det'(\Phi)$ is independent of the choice of deleted rows and columns. □

Similar property holds for $\mathrm{Pf}'(\Psi)$:

Proposition A.2: *The reduced Pfaffian* $\mathrm{Pf}'(\Psi)$ *is independent of the choice on which two rows and columns being deleted, as long as they are in the range* 1 *to* n.

Proof of Proposition A.2. The proof is carried out in the same manner as the previous one. It is sufficient to just prove that

$$\frac{1}{\sigma_{ij}}\mathrm{Pf}\left(\Psi^{ij}_{ij}\right) = -\frac{1}{\sigma_{i,j+1}}\mathrm{Pf}\left(\Psi^{i,j+1}_{i,j+1}\right) \qquad (1 < i < j \leqslant n-1)\,. \tag{A.2}$$

Then we multiply $\sigma_{i,j+1}$ to the $(j-1)$-th row and column of Ψ^{ij}_{ij}, and similarly multiply σ_{ij} to the $(j-1)$-th row and column of $\Psi^{i,j+1}_{i,j+1}$. The new matrices obtained this way are called Ψ' and Ψ'', and their Pfaffians equal to each other:

$$\mathrm{Pf}\left(\Psi'\right) = \sigma_{i,j+1}\mathrm{Pf}\left(\Psi^{ij}_{ij}\right) \qquad \mathrm{Pf}\left(\Psi''\right) = \sigma_{ij}\mathrm{Pf}\left(\Psi^{i,j+1}_{i,j+1}\right).$$

Then Ψ' and Ψ'' only differ by the $(j-1)$-th row and column. Since both of them are antisymmetric, we only display the rows explicitly:

$$\begin{aligned} &[\Psi']_{j-1}: && \sigma_{i,j+1} \times \left(\;\; \boldsymbol{a}_{j+1} \;\; \middle| \;\; \boldsymbol{c}_{j+1} \;\; \right) \\ &[\Psi'']_{j-1}: && \sigma_{ij} \times \left(\;\; \boldsymbol{a}_{j} \;\; \middle| \;\; \boldsymbol{c}_{j} \;\; \right), \end{aligned}$$

where $\boldsymbol{a}_j$ and $\boldsymbol{a}_{j+1}$ are $(n-2)$-component row vectors:

$$\begin{pmatrix} \boldsymbol{a}_j \\ \boldsymbol{a}_{j+1} \end{pmatrix} = \begin{pmatrix} \frac{s_{j1}}{\sigma_{j1}} & \frac{s_{j2}}{\sigma_{j2}} & \cdots & \widehat{\frac{s_{ji}}{\sigma_{ji}}} & \cdots & \frac{s_{j,j-1}}{\sigma_{j,j-1}} & \frac{s_{j,j+1}}{\sigma_{j,j+1}} & \frac{s_{j,j+2}}{\sigma_{j,j+2}} & \cdots & \frac{s_{jn}}{\sigma_{jn}} \\ \frac{s_{j+1,1}}{\sigma_{j+1,1}} & \frac{s_{j+1,2}}{\sigma_{j+1,2}} & \cdots & \frac{s_{j+1,i}}{\sigma_{j+1,i}} & \cdots & \frac{s_{j+1,j-1}}{\sigma_{j+1,j-1}} & \frac{s_{j+1,j}}{\sigma_{j+1,j}} & \frac{s_{j+1,j+2}}{\sigma_{j+1,j+2}} & \cdots & \frac{s_{j+1,n}}{\sigma_{j+1,n}} \end{pmatrix}.$$

Meanwhile, $\boldsymbol{c}_j$ and $\boldsymbol{c}_{j+1}$ are n-component row vectors:

$$\begin{pmatrix} \boldsymbol{c}_j \\ \boldsymbol{c}_{j+1} \end{pmatrix} = \begin{pmatrix} \frac{(\epsilon k)_{1j}}{\sigma_{1j}} & \cdots & \frac{(\epsilon k)_{j-1,j}}{\sigma_{j-1,j}} & \Sigma_j & \frac{(\epsilon k)_{j+1,j}}{\sigma_{j+1,j}} & \frac{(\epsilon k)_{j+2,j}}{\sigma_{j+2,j}} & \cdots \\ \frac{(\epsilon k)_{1,j+1}}{\sigma_{1,j+1}} & \cdots & \frac{(\epsilon k)_{j-1,j+1}}{\sigma_{j-1,j+1}} & \frac{(\epsilon k)_{j,j+1}}{\sigma_{j,j+1}} & \Sigma_{j+1} & \frac{(\epsilon k)_{j+2,j+1}}{\sigma_{j+2,j+1}} & \cdots \end{pmatrix},$$

where we have used the shorthand notation:

$$(\epsilon k)_{ab} = -2\epsilon_a \cdot k_b \qquad \Sigma_a = \sum_{b\neq a} \frac{2\epsilon_a \cdot k_b}{\sigma_{ab}} = -\sum_{b\neq a} \frac{(\epsilon k)_{ab}}{\sigma_{ab}}\,.$$

Now in Ψ', we multiply the row started with $\frac{s_{k1}}{\sigma_{k1}}$ by σ_{ik}, adding it to the $(j-1)$-th row, and then do the same to the columns. The result is:

- The $\boldsymbol{a}$ part becomes:

$$\sigma_{i,j+1}\frac{s_{j+1,l}}{\sigma_{j+1,l}} + \sum_{k\neq i,j,j+1,l} \sigma_{ik}\frac{s_{kl}}{\sigma_{kl}} = \sigma_{il}\sum_{k\neq j,i,l}\frac{s_{kl}}{\sigma_{kl}} - \sum_{k\neq j,i,l} s_{kl} = -\sigma_{il}\frac{s_{jl}}{\sigma_{jl}} + s_{jl} = -\sigma_{ij}\frac{s_{jl}}{\sigma_{jl}}\,.$$

  Namely, we have $\sigma_{i,j+1} \times \boldsymbol{a}_{j+1} \to -\sigma_{ij} \times \boldsymbol{a}_j$.

- Now we study the $\boldsymbol{c}$ part. For the column containing $(\epsilon k)_{j,j+1}/\sigma_{j,j+1}$, there is no diagonal elements in the summation, such that we have:

$$-\sigma_{i,j+1}\frac{\epsilon_j \cdot k_{j+1}}{\sigma_{j,j+1}} - \sum_{p\neq i,j,j+1}\sigma_{ip}\frac{2\epsilon_j \cdot k_p}{\sigma_{j,p}} = -\sigma_{ij}\sum_{p\neq i,j}\frac{2\epsilon_j \cdot k_p}{\sigma_{jp}} + 2\epsilon_j \cdot k_i = -\sigma_{ij}\Sigma_j\,.$$

  For the other columns with label l, there must be one diagonal element Σ_l involved in the summations, such that the result turns out to be:

$$\sigma_{il}\Sigma_l - \sum_{p\neq i,j,l}\sigma_{ip}\frac{2\epsilon_l \cdot k_p}{\sigma_{lp}} = \sigma_{il}\frac{2\epsilon_l \cdot k_j}{\sigma_{lj}} - \sum_{p\neq j,l}2\epsilon_l \cdot k_p = \sigma_{il}\frac{2\epsilon_l \cdot k_j}{\sigma_{lj}} + 2\epsilon_l \cdot k_j = \sigma_{ij}\frac{2\epsilon_l \cdot k_j}{\sigma_{lj}}\,.$$

  Therefore, we have $\sigma_{i,j+1} \times \boldsymbol{c}_{j+1} \to -\sigma_{ij} \times \boldsymbol{c}_j$ in the $\boldsymbol{c}$ part.

- The $(j-1)$-th column transforms identically as the row.

After this operation, we find that Ψ' and Ψ'' are identical except for a minus sign in the $(j-2)$-th row and column. Since the elementary transformation does not change the Pfaffian, we must have:

$$\mathrm{Pf}\,(\Psi') = -\mathrm{Pf}\,(\Psi'')\,,$$

such that Eq. (A.2) follows immediately. In this way, we have proved that $\mathrm{Pf}'(\Psi)$ does not depend on the choice of the deleted rows and the corresponding columns. □

With these two propositions being proved, it is very straightforward to see that both $\det'(\Phi)$ and $\mathrm{Pf}'(\Psi)$ are invariant under permutations. Suppose we exchange two particles a and b, if neither of them belongs to those deleted rows and columns, the permutation amounts to exchange a pair of rows and columns in Φ, which leaves the determinant invariant; if any of a and b coincides with the deleted rows and columns, the permutation amounts to choose another set of deleted rows or columns, which again leaves $\det'(\Phi)$ invariant due to Proposition A.1. Exactly the same argument holds to show that $\mathrm{Pf}'(\Psi)$ is invariant. The only difference is that we now need to exchange two pairs of rows and columns, since each particle index appears twice in Ψ.

APPENDIX B

THE PARKE-TAYLOR FACTOR AND AMPLITUDE RELATIONS

As demonstrated in Appendix A, both $\det'(\Phi)$ and $\mathrm{Pf}'(\Psi)$ in the Yang-Mills integrand are invariant under permutatioins, such that the only ingredient that can encode the amplitude relations is the Parke-Taylor factor (2.137), repeated here as:

$$PT(\boldsymbol{I}) = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \equiv \frac{1}{(12)(23)\ldots(n1)}\,, \tag{B.1}$$

where we have further simplified the notation by using $(ab) \equiv \sigma_{ab}$. The main purpose of this appendix is to verify that PT indeed manifestly satisfies all the amplitude relations listed in Section 2.2.3.

First, the cyclic symmetry and reflection are trivially satisfied as complex number identities:

$$PT(12\ldots n) = PT(2\ldots n1) \qquad PT(12\ldots n) = (-1)^n PT(n\ldots 21)\,. \tag{B.2}$$

Namely, we do not need to require $\{\sigma\}$ be a solution to the SE (3.5). Less trivially, the KK relation is also just a complex number identity of PT, with no requirement on $\{\sigma\}$:

Theorem B.1: *PT satisfies the KK relation:*

$$(-1)^{|\boldsymbol{\beta}|}PT(1,\boldsymbol{\alpha},n,\boldsymbol{\beta}) = \sum_{\boldsymbol{\sigma}\in OP(\boldsymbol{\alpha},\boldsymbol{\beta}^T)} PT(1,\boldsymbol{\sigma},n) \tag{B.3}$$

for any set of complex numbers $\{\sigma\}$

Next, we demonstrate this point by two explicit 6-point calculations. The general n-point proof will be given later.

Example 1: We first choose $\boldsymbol{\alpha} = \{2,3,4\}$ and $\boldsymbol{\beta} = \{5\}$. When $|\boldsymbol{\beta}| = 1$, the KK relation reduces to the $U(1)$ decoupling identity. The left hand side of Eq. (B.2) becomes:

$$-PT(123465) = -\frac{1}{(12)(23)(34)(46)(65)(51)} = -PT(12346) \times \frac{(61)}{(65)(51)}\,.$$

Next, we rewrite the last factor as:[1]

$$\frac{(ab)}{(ac)(cb)} = \int_{\sigma_a}^{\sigma_b} \frac{dw}{(w-\sigma_c)^2} \equiv \int_{(a)}^{(b)} \frac{dw}{(wc)^2}\,, \tag{B.4}$$

such that we have:

$$-PT(123465) = -PT(12346) \int_{(6)}^{(1)} \frac{dw}{(w5)^2} = PT(12346) \int_{(1)}^{(6)} \frac{dw}{(w5)^2}\,. \tag{B.5}$$

For this case, the right hand side of Eq. (B.2) reads (we underline the numbers in $\boldsymbol{\beta}$):

$$\begin{aligned} &PT(1234\underline{5}6) + PT(123\underline{5}46) + PT(12\underline{5}346) + PT(1\underline{5}2346) \\ &= PT(12346)\left[\frac{(46)}{(45)(56)} + \frac{(34)}{(35)(54)} + \frac{(23)}{(25)(53)} + \frac{(12)}{(15)(52)}\right] \\ &= PT(12346)\left[\int_{(4)}^{(6)} + \int_{(3)}^{(4)} + \int_{(2)}^{(3)} + \int_{(1)}^{(2)}\right] \frac{dw}{(w5)^2} \\ &= PT(12346) \int_{(1)}^{(6)} \frac{dw}{(w5)^2}\,, \end{aligned} \tag{B.6}$$

which indeed equals Eq. (B.5). We do not need to specify the integration path since the integrand has no simple pole.

Example 2: Next, we consider $\boldsymbol{\alpha} = \{2,3\}$ and $\boldsymbol{\beta} = \{5,4\}$. This is a less trivial but more illustrative example. The left hand side of Eq. (B.2) becomes:

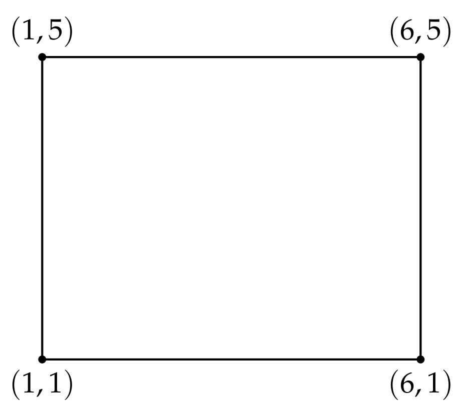
$$PT(1236\underline{54}) = PT(1236)\frac{(61)}{(65)(51)}\frac{(51)}{(54)(41)} = PT(1236)\int_{(1,1)}^{(6,5)} \Omega(54)\,, \tag{B.7}$$

where the 2-form $\Omega(54)$ is:

$$\Omega(54) = \frac{dw_1 dw_2}{(w_1 5)^2 (w_2 4)^2}\,. \tag{B.8}$$

In particular, the integration area is the cube with principal diagonal $(1,1) \equiv (\sigma_1, \sigma_1)$ to $(6,5) \equiv (\sigma_6, \sigma_5)$ in the (w_1, w_2) space:

$$\int_{(1,1)}^{(6,5)} = \tag{B.9}$$



[1] This identity is first used by Hodges to prove the cancellation of spurious poles in the BCFW results [24].

Now the right hand side of Eq. (B.2) becomes:

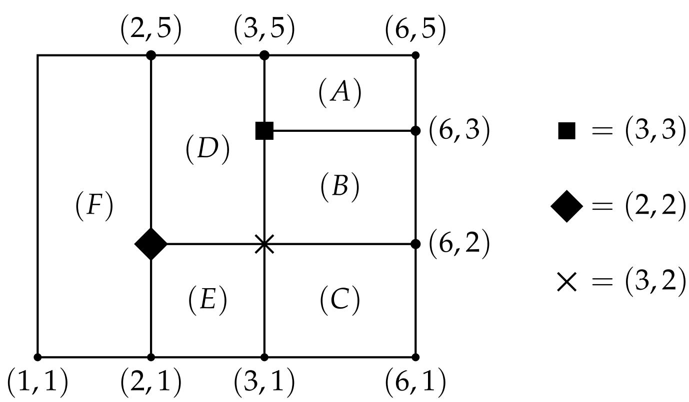
$$
\begin{aligned}
& PT(123\underline{45}6) + PT(12\underline{4}3\underline{5}6) + PT(1\underline{4}23\underline{5}6) \\
+ & PT(12\underline{45}36) + PT(1\underline{4}2\underline{5}36) \\
+ & PT(1\underline{45}236)\,.
\end{aligned}
\tag{B.10}
$$

The reason for such an arrangement will be clear soon. Each of these six terms corresponds to an integration of $\Omega(54)$:

$$
\begin{aligned}
&(A) \qquad && PT(123\underline{45}6) = PT(1236) \int_{(3,3)}^{(6,5)} \Omega(54) \\
&(B) \qquad && PT(12\underline{4}3\underline{5}6) = PT(1236) \int_{(3,2)}^{(6,3)} \Omega(54) \\
&(C) \qquad && PT(1\underline{4}23\underline{5}6) = PT(1236) \int_{(3,1)}^{(6,2)} \Omega(54) \\
&(D) \qquad && PT(12\underline{45}36) = PT(1236) \int_{(2,2)}^{(3,5)} \Omega(54) \\
&(E) \qquad && PT(1\underline{4}2\underline{5}36) = PT(1236) \int_{(2,1)}^{(3,2)} \Omega(54) \\
&(F) \qquad && PT(1\underline{45}236) = PT(1236) \int_{(1,1)}^{(2,5)} \Omega(54)\,.
\end{aligned}
$$

The total integration region is exactly (B.9):

$$\int_{A+B+C+D+E+F} =$$



(B.11)

which establishes the equality between Eq. (B.7) and Eq. (B.10). We note that each line in Eq. (B.10) gives a vertical stripe in Eq. (B.11).

Now we are ready for generic cases. Suppose we have $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \dots, \beta_r\}$ with $r + s = n - 2$, the two examples above inspire us to define the following r-form:

$$\Omega(\beta_1\beta_2 \dots \beta_r) \equiv \prod_{i=1}^{r} \frac{dw_i}{(\omega_i\beta_i)^2}\,. \tag{B.12}$$

Then the left hand side of Eq. (B.2) just corresponds to the integration of Ω over a cube with principal diagonal $(1, 1 \dots 1)$ to $(n, \beta_1, \beta_2 \dots \beta_{r-1})$ in the space of $(\omega_1, \omega_2 \dots \omega_r)$:

$$\begin{aligned}(-1)^{|\boldsymbol{\beta}|} PT(1, \boldsymbol{\alpha}, n, \boldsymbol{\beta}) &= PT(1, \boldsymbol{\alpha}, n) \frac{(n1)}{(n\beta_1)(\beta_1 1)} \frac{(\beta_1 1)}{(\beta_1\beta_2)(\beta_2 1)} \cdots \frac{(\beta_{r-1} 1)}{(\beta_{r-1}\beta_r)(\beta_r 1)} \\ &= PT(1, \boldsymbol{\alpha}, n) \int_{(1,1\dots 1)}^{(n, \beta_1 \dots \beta_{r-1})} \Omega(\beta_1\beta_2 \dots \beta_r)\,, \end{aligned} \tag{B.13}$$

where the $(-1)^{|\boldsymbol{\beta}|}$ factor is absorbed by a change of the integral orientation. Again there is no need to specify the integration path since Ω has no simple pole. Therefore, to prove Theorem B.1, we only need to show that the right hand side of Eq. (B.2) gives the same integration. This can be done with the help of the following lemma, to be proved later:

Lemma B.1: *If we have a string $\{\beta_r, \beta_{r-1} \dots \beta_p\}$ strictly before α_i,[2] then*

$$\sum_{\boldsymbol{\sigma}} PT(1, \boldsymbol{\sigma}, \alpha_i \dots) = PT(1, \boldsymbol{\alpha}_i \dots) \int_{(\alpha_{i-1}, 1 \dots 1)}^{(\alpha_i, \beta_p \dots \beta_{r-1})} \Omega(\beta_p \dots \beta_r)\,, \tag{B.14}$$

where $\boldsymbol{\alpha}_i = \{\alpha_1 \dots \alpha_{i-1}, \alpha_i\}$. The summation of $\boldsymbol{\sigma}$ is over the order preserved permutations:

$$\boldsymbol{\sigma} \in OP\,(\boldsymbol{\alpha}_{i-1}, \widetilde{\boldsymbol{\beta}}_{p+1}) \cup \{\beta_p\}\,, \qquad \boldsymbol{\alpha}_{i-1} = \{\alpha_1 \dots \alpha_{i-1}\} \qquad \widetilde{\boldsymbol{\beta}}_{p+1} = \{\beta_r \dots \beta_{p+1}\}\,,$$

namely, $\boldsymbol{\sigma}$ must have β_p come last, such that β_p must be right before α_i. If $i = 1$, we just define $\alpha_0 \equiv 1$. Finally, if $p = r$, then the summation reduces to only one term $\boldsymbol{\sigma} = \{\alpha_1 \dots \alpha_{i-1}, \beta_p\}$ ($\boldsymbol{\sigma} = \{\beta_p\}$ if $i = 1$).

We now present the proof of Theorem B.1 with the help of Lemma B.1:

Proof of Theorem B.1. In Eq. (B.14), we choose $p = 1$:

$$\sum_{\widetilde{\boldsymbol{\sigma}}} PT(1, \widetilde{\boldsymbol{\sigma}}, \alpha_i \dots \alpha_s, n) = PT(1, \boldsymbol{\alpha}, n) \int_{(\alpha_{i-1}, 1 \dots 1)}^{(\alpha_i, \beta_1 \dots \beta_{r-1})} \Omega(\beta_1 \dots \beta_r)\,, \tag{B.15}$$

where $\widetilde{\boldsymbol{\sigma}} \in \mathrm{OP}(\boldsymbol{\alpha}_{i-1}, \widetilde{\boldsymbol{\beta}}_2) \cup \{\beta_1\}$. It is easy to realize that Eq. (B.15) gives the integration over a stripe with principal diagonal $(\alpha_{i-1}, 1 \dots 1)$ to $(\alpha_i, \beta_1 \dots \beta_{r-1})$. In Eq. (B.11), the three vertical stripes $(A + B + C)$, $(D + E)$ and (F) are just special examples of our general

[2] The "strictly before" means that there must be elements in $\{\beta_p \dots \beta_r\}$ between α_i and α_{i-1}. It is equivalent to putting β_p right before α_i.

expresseion (B.15). Finally, to sum over $\boldsymbol{\sigma} \in \mathrm{OP}(\boldsymbol{\alpha}, \boldsymbol{\beta}^T)$, we just need to sum over all possible positions that $\boldsymbol{\beta}^T$ can be strictly before:

$$\begin{aligned}
\sum_{\boldsymbol{\sigma} \in \mathrm{OP}(\boldsymbol{\alpha}, \boldsymbol{\beta}^T)} PT(1, \boldsymbol{\sigma}, n) &= \sum_{i=1}^{s+1} \sum_{\widetilde{\boldsymbol{\sigma}}} PT(1, \widetilde{\boldsymbol{\sigma}}, \alpha_i \dots \alpha_s, n) \\
&= PT(1, \boldsymbol{\alpha}, n) \sum_{i=1}^{s+1} \int_{(\alpha_{i-1}, 1 \dots 1)}^{(\alpha_i, \beta_1 \dots \beta_{r-1})} \Omega(\beta_1 \dots \beta_r) \\
&= PT(1, \boldsymbol{\alpha}, n) \int_{(1 \dots 1)}^{(n, \beta_1 \dots \beta_{r-1})} \Omega(\beta_1 \dots \beta_r) \,, \qquad \text{(B.16)}
\end{aligned}$$

where we have identified that $\alpha_0 \equiv 1$ and $\alpha_{s+1} \equiv n$. Again Eq. (B.11) is a good way to visualize how these stripes add up to the cubic region we want. Since Eq. (B.16) exactly equals Eq. (B.13), Theorem B.1 is thus proved. □

Finally, we come to prove Lemma B.1. The following result will be used in the proof: if a consecutive string of β's is sandwiched between two adjacent α's, we have:

$$\begin{aligned}
& PT(\dots \alpha_{i-1}, \beta_q \dots \beta_p, \alpha_i \dots) \\
&= PT(\dots \alpha_{i-1}, \alpha_i \dots) \frac{(\alpha_{i-1}\beta_{q-1})}{(\alpha_{i-1}\beta_q)(\beta_q\beta_{q-1})} \frac{(\alpha_{i-1}\beta_{q-2})}{(\alpha_{i-1}\beta_{q-1})(\beta_{q-1}\beta_{q-2})} \cdots \frac{(\alpha_{i-1}\alpha_i)}{(\alpha_{i-1}\beta_p)(\beta_p\alpha_i)} \\
&= PT(\dots \alpha_{i-1}, \alpha_i \dots) \int_{(\alpha_{i-1}, \alpha_{i-1} \cdots\, \alpha_{i-1})}^{(\alpha_i, \beta_p \cdots\, \beta_{q-1})} \Omega(\beta_p \beta_{p+1} \dots \beta_q) \,. \qquad \text{(B.17)}
\end{aligned}$$

Then Lemma B.1 can be proved using induction:

Proof of Lemma B.1. We first look at the starting point of the induction. For $r - p = 0$, Eq. (B.14) reads:

$$PT(1, \boldsymbol{\alpha}_{i-1}, \beta_r, \alpha_i \dots) = PT(1, \boldsymbol{\alpha}_i \dots) \int_{(\alpha_{i-1})}^{(\alpha_i)} \Omega(\beta_r) \,.$$

For $r - p = 1$, Eq. (B.14) reads:

$$\begin{aligned}
& \sum_{j=1}^{i-1} PT(1, \boldsymbol{\alpha}_{j-1}, \beta_r, \alpha_j \dots \beta_{r-1}, \alpha_i \dots) + PT(1, \boldsymbol{\alpha}_{i-1}, \beta_r, \beta_{r-1}, \alpha_i \dots) \\
&= PT(1, \boldsymbol{\alpha}_i \dots) \frac{(\alpha_{i-1}\alpha_i)}{(\alpha_{i-1}\beta_{r-1})(\beta_{r-1}\alpha_i)} \left[\sum_{j=1}^{i-1} \frac{(\alpha_{j-1}\alpha_j)}{(\alpha_{j-1}\beta_r)(\beta_r\alpha_j)} + \frac{(\alpha_{i-1}\beta_{r-1})}{(\alpha_{i-1}\beta_r)(\beta_r\beta_{r-1})} \right] \\
&= PT(1, \boldsymbol{\alpha}_i \dots) \int_{(\alpha_{i-1}, 1)}^{(\alpha_i, \beta_{r-1})} \Omega(\beta_{r-1}\beta_r) \,.
\end{aligned}$$

Therefore, Eq. (B.14) has been proved for the first two cases.

Now we assume that Eq. (B.14) holds up to an arbitrary $r-p$. In other words, Eq. (B.14) holds for any $\widetilde{\boldsymbol{\beta}}$ shorter or equal to $\widetilde{\boldsymbol{\beta}}_{p+1}$. Then we examine the case of $r-p+1$. The problem is thus to sum over all order preserved permutations with the string $\{\beta_r,\ldots,\beta_{p-1}\}$ strictly before α_i, namely, β_{p-1} is right before α_i:

$$\sum_{\boldsymbol{\sigma}} PT(1,\boldsymbol{\sigma},\alpha_i\ldots) \qquad \boldsymbol{\sigma}\in \mathrm{OP}\,(\boldsymbol{\alpha}_{i-1},\widetilde{\boldsymbol{\beta}}_p)\cup\{\beta_{p-1}\}\,.$$

The permutation $\{1,\boldsymbol{\sigma},\alpha_i\}$ can be further divided according to:

1. How many β's are between α_i and α_{i-1};
2. The α_k that the rest β's are strictly before.

We can thus rewrite the summation over $\boldsymbol{\sigma}$ as:

$$\sum_{\boldsymbol{\sigma}} PT(1,\boldsymbol{\sigma},\alpha_i\ldots) = \sum_{j=p-1}^{r}\sum_{k=1}^{i-1}\sum_{\widetilde{\boldsymbol{\sigma}}} PT(1,\widetilde{\boldsymbol{\sigma}},\alpha_k\ldots\alpha_{i-1},\beta_j\ldots\beta_p,\beta_{p-1},\alpha_i\ldots)\,, \tag{B.18}$$

where $\widetilde{\boldsymbol{\sigma}}\in \mathrm{OP}\,(\boldsymbol{\alpha}_{k-1},\widetilde{\boldsymbol{\beta}}_{j+2})\cup\{\beta_{j+1}\}$. By construction, there is no β between α_k and α_{i-1}, and there is no α between β_j and β_{p-1}. For $j=r-1$, the summation over $\widetilde{\boldsymbol{\sigma}}$ contains only one term $\widetilde{\boldsymbol{\sigma}}=\{\alpha_1\ldots\alpha_{k-1},\beta_r\}$ ($\widetilde{\boldsymbol{\sigma}}=\{\beta_r\}$ if $k=1$). Similarly, for $k=1$, the summation over $\widetilde{\boldsymbol{\sigma}}$ also reduces to only $\widetilde{\boldsymbol{\sigma}}=\{\beta_r\ldots\beta_{j+1}\}$. Finally, for $j=r$, the double summation over k and $\widetilde{\boldsymbol{\sigma}}$ contains only the single term:

$$PT(1,\boldsymbol{\alpha}_{i-1},\beta_r\ldots\beta_p,\beta_{p-1},\alpha_i\ldots)\,.$$

In any situation, the length of $\widetilde{\boldsymbol{\beta}}_{j+2}$ is at most that of $\widetilde{\boldsymbol{\beta}}_{p+1}$, such that we can use the induction assumption, together with Eq. (B.17), to write:

$$\begin{aligned}
&\sum_{k=1}^{i-1}\sum_{\widetilde{\boldsymbol{\sigma}}} PT(1,\widetilde{\boldsymbol{\sigma}},\alpha_k\ldots\alpha_{i-1},\beta_j\ldots\beta_p,\beta_{p-1},\alpha_i\ldots)\\
&= PT(1,\boldsymbol{\alpha}_i\ldots)\int_{(\alpha_{i-1}\ldots\,\alpha_{i-1})}^{(\alpha_i,\,\beta_{p-1}\ldots\,\beta_{j-1})}\Omega(\beta_{p-1}\ldots\beta_j)\sum_{k=1}^{i-1}\int_{(\alpha_{k-1},\,1\,\ldots\,1)}^{(\alpha_k,\,\beta_{j+1}\ldots\,\beta_{r-1})}\Omega(\beta_{j+1}\ldots\beta_r)\\
&= PT(1,\boldsymbol{\alpha}_i\ldots)\int_{(\alpha_{i-1}\ldots\,\alpha_{i-1}|1\,\ldots\,1)}^{(\alpha_i,\,\beta_{p-1}\ldots\,\beta_{j-1}|\alpha_{i-1},\,\beta_{j+1}\ldots\,\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\\
&\equiv PT(1,\boldsymbol{\alpha}_i\ldots)\mathcal{F}(j)\,.
\end{aligned}\tag{B.19}$$

The integration is taken in an $r-p+2$ dimensional cube, and the vertical line separates the first $j-p+2$ and the last $r-j$ coordinates. The last step is to take the j-sum over the $\mathcal{F}(j)$. This can be done using another induction. First, we have:

$$\mathcal{F}(r)+\mathcal{F}(r-1) = \left[\int_{(\alpha_{i-1}\ldots\alpha_{i-1},\alpha_{i-1})}^{(\alpha_i,\beta_{p-1}\ldots\beta_{r-2},\beta_{r-1})} + \int_{(\alpha_{i-1}\ldots\alpha_{i-1},1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{r-2},\alpha_{i-1})}\right]\Omega(\beta_{p-1}\ldots\beta_r)$$
$$= \int_{(\alpha_{i-1}\ldots\alpha_{i-1},1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{r-2},\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\,.$$

Now if the sum is taken from $j=m$, we assume that:

$$\sum_{j=m}^{r}\mathcal{F}(j) = \int_{(\alpha_{i-1}\ldots\alpha_{i-1}|1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{m-1}|\beta_m\ldots\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\,,$$

where the vertical line separates the first $m-p+2$ coordinates and the last $r-m$ coordinates. Then adding $\mathcal{F}(m-1)$ gives:

$$\sum_{j=m-1}^{r}\mathcal{F}(j) = \left[\int_{(\alpha_{i-1}\ldots\alpha_{i-1}|1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{m-1}|\beta_m\ldots\beta_{r-1})} + \int_{(\alpha_{i-1}\ldots\alpha_{i-1},1|1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{m-2},\alpha_{i-1}|\beta_m\ldots\beta_{r-1})}\right]\Omega(\beta_{p-1}\ldots\beta_r)$$
$$= \int_{(\alpha_{i-1}\ldots\alpha_{i-1},1|1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{m-2},\beta_{m-1}|\beta_m\ldots\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\,,$$

namely, in the lower limit the length of 1's grows by one to the left. Therefore, by induction, we can write down the result:

$$\sum_{j=p-1}^{r}\mathcal{F}(j) = \int_{(\alpha_{i-1},1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\,. \tag{B.20}$$

Plugging it into Eq. (B.19) and then Eq. (B.18), we get the final result:

$$\sum_{\boldsymbol{\sigma}} PT(1,\boldsymbol{\sigma},\alpha_i\ldots) = PT(1,\boldsymbol{\alpha}_i\ldots)\int_{(\alpha_i,1\ldots1)}^{(\alpha_i,\beta_{p-1}\ldots\beta_{r-1})}\Omega(\beta_{p-1}\ldots\beta_r)\,, \tag{B.21}$$

where $\boldsymbol{\sigma} = \mathrm{OP}\,(\boldsymbol{\alpha}_{i-1},\widetilde{\boldsymbol{\beta}}_p)\cup\{\beta_{p-1}\}$. Eq. (B.14) and the Lemma B.1 is thus proved by the principle of induction. □

At last, we show that PT also satisfies the BCJ relation:

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i}s_{j2}\right)PT(1\ldots i,2,i+1\ldots n) = 0\,. \tag{B.22}$$

Unlike the previous cases, the BCJ relation does not hold for arbitrary σ's as a complex number identity. It is true if and only if $\{\sigma\}$ is a solution to the SE. Namely, the n-particle kinematics is involved. We present the proof as the following:

Proof of the BCJ relation: We can prove the relation by directly rewrite the left hand side of Eq. (B.22) as:

$$\begin{aligned}
&\sum_{i=3}^{n}\left(\sum_{j=3}^{i} s_{j2}\right) PT(1\dots i,2,i+1\dots n) \\
&= \sum_{i=3}^{n}\sum_{j=3}^{i} \frac{s_{j2}}{(13)(34)\dots(i2)(2,i+1)\dots(n1)} \\
&= PT(134\dots n)\sum_{i=3}^{n}\sum_{j=3}^{i} \frac{s_{j2}(i,i+1)}{(i2)(2,i+1)} \\
&= PT(134\dots n)\sum_{i=3}^{n}\sum_{j=3}^{i} s_{j2}\int_{(i)}^{(i+1)} \frac{dw}{(w2)^2}\,.
\end{aligned} \tag{B.23}$$

We then can change the order of summation: $\sum_{i=3}^{n}\sum_{j=3}^{i} = \sum_{j=3}^{n}\sum_{i=j}^{n}$, such that:

$$\begin{aligned}
\sum_{i=3}^{n}\sum_{j=3}^{i} s_{j2}\int_{(i)}^{(i+1)} \frac{dw}{(w2)^2} &= \sum_{j=3}^{n}\sum_{i=j}^{n} s_{j2}\int_{(i)}^{(i+1)} \frac{dw}{(w2)^2} \\
&= \sum_{j=3}^{n} s_{j2}\int_{(j)}^{(1)} \frac{dw}{(w2)^2} \\
&= \sum_{j=3}^{n} \frac{s_{j2}\sigma_{j1}}{\sigma_{j2}\sigma_{21}}\,.
\end{aligned} \tag{B.24}$$

Using $\sigma_{j1} = \sigma_{j2} + \sigma_{21}$, we can easily verify that:

$$\sum_{j=3}^{n} \frac{s_{j2}\sigma_{j1}}{\sigma_{j2}\sigma_{21}} = \sum_{j=3}^{n} \frac{s_{j2}}{\sigma_{21}} + \sum_{j=3}^{n} \frac{s_{j2}}{\sigma_{j2}} = \sum_{j=1,\, j\neq 2}^{n} \frac{s_{j2}}{\sigma_{j2}} = 0\,. \tag{B.25}$$

Therefore, we have proved that the BCJ relation:

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i} s_{j2}\right) PT(1\dots i,2,i+1\dots n) = 0 \tag{B.26}$$

holds if and only if the $\{\sigma\}$ satisfies the SE. Actually, this relation, first noticed by Cachazo in [31], motivated the establishment of the CHY formalism. □

APPENDIX C

EULERIAN SECTORS

In this appendix, we are going to show that in four dimensions, the solutions to SE fall into sectors. The number of solutions in each sector displays an Eulerian number pattern. The 4d spacetime is critical here since we need the spinor helicity formalism to derive this property. The proof is carried out using induction at the soft limit, like the one used to prove that there are in all $(n-3)!$ solutions to SE shown in Section 3.1.1.

We start with the map $P^\mu(z)$ defined in Eq. (3.3):

$$P^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z-\sigma_a} = \frac{\sum_{a=1}^{n} k_a^\mu \prod_{b\neq a}(z-\sigma_b)}{\prod_{a=1}^{n}(z-\sigma_a)} \,. \tag{C.1}$$

Due to the momentum conservation, the numerator of $P^\mu(z)$ is a degree $n-2$ polynomial in z. As discussed in Section 3.1, the SE (3.5) is equivalent to imposing the null condition $P^2(z) = 0$ on the entire Riemann sphere. In four dimensions, we can equivalently impose this null condition by requiring that

$$P^{\alpha\dot{\alpha}}(z) = \frac{\lambda^\alpha(z)\widetilde{\lambda}^{\dot{\alpha}}(z)}{\prod_{a=1}^{n}(z-\sigma_a)} \,, \tag{C.2}$$

where the degree of $\lambda(z)$ and $\widetilde{\lambda}(z)$ should add to $n-2$:

$$\deg[\lambda_\alpha(z)] = d \qquad \deg[\widetilde{\lambda}_{\dot{\alpha}}(z)] = n-2-d \qquad d \in \{1, 2 \ldots n-3\} \,. \tag{C.3}$$

These two polynomials, together with the puncture positions $\{\sigma\}$, can be determined by requiring that the residue of $P^\mu(z)$ at σ_a equals k_a^μ:

$$(k_a)^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha(\sigma_a)\widetilde{\lambda}^{\dot{\alpha}}(\sigma_a)}{\prod_{b\neq a}(\sigma_a-\sigma_b)} \,. \tag{C.4}$$

The unknowns of Eq. (C.4) are the σ's, and the coefficients of $\lambda(z)$ and $\widetilde{\lambda}(z)$:[1]

$$\lambda_\alpha(z) = \sum_{m=0}^{d} \rho_{\alpha m} z^m \qquad \widetilde{\lambda}_{\dot{\alpha}}(z) = \sum_{m=0}^{n-2-d} \widetilde{\rho}_{\dot{\alpha} m} z^m \,. \tag{C.5}$$

---

[1]For generic external momentum data, $d=1$ or $d=n-2$ are NOT allowed. This would only lead to $k_a \cdot k_b = 0$ for any pairs, according to Eq. (C.4).

In all, there are $2(d+1)+2(n-1-d)-1 \;=\; 2n-1$ independent ρ and $\widetilde{\rho}$, taking into account the rescaling freedom in λ and $\widetilde{\lambda}$. Together with $n-3$ independent σ's after the $SL(2,\mathbb{C})$ gauge fixing, the total number of independent unknowns is $3n-4$. On the other hand, there are in all $4n$ equations in Eq. (C.4), in which only $3n-4$ are independent.[2] Since the number of unknowns exactly matches the number of equations, the unknowns $\{\sigma,\rho,\widetilde{\rho}\}$ always have solutions. In addition, the set $\{\sigma\}$ solved from Eq. (C.4) must agree with that solved from the original SE (3.5). In Section 3.1.1, we proved that the SE has $(n-3)!$ solutions in $\{\sigma\}$. Since we expect that Eq. (C.4) is equivalent to the original form (3.5), we need to count that Eq. (C.4) also has $(n-3)!$ solutions. As we are going to show below, this counting naturally put the solutioins into sectors labelled by the degree of the polynomial $\lambda(z)$, or equivalently, the degree of $\widetilde{\lambda}(z)$.

We first look at those solutions that make $\deg[\lambda(z)] = d$. We assume that the number of such solutions is $\mathcal{N}_{n,d}$. In the soft limit $k_n^\mu \to 0$, we have:

$$(k_n)^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha(\sigma_n)\widetilde{\lambda}^{\dot{\alpha}}(\sigma_n)}{\prod_{b=1}^{n-1}(\sigma_n-\sigma_b)} \;\to\; 0\,. \tag{C.6}$$

This means we either have $\lambda^\alpha(\sigma_n)=0$ or $\widetilde{\lambda}^{\dot{\alpha}}(\sigma_n)=0$. For the first case, we can parameterize $\lambda^\alpha(z)$ at the soft limit as:

$$\lambda^\alpha(z) \;\to\; (z-\sigma_n)\lambda_\star^\alpha(z) \qquad \det[\lambda_\star^\alpha(z)] = d-1\,. \tag{C.7}$$

If we plug this into the other equations, the outstanding factor $(z-\sigma_n)$ will get cancelled by the same factor in the denominator, such that we get:

$$(k_a)^{\alpha\dot{\alpha}} = \frac{\lambda^\alpha(\sigma_a)\widetilde{\lambda}^{\dot{\alpha}}(\sigma_a)}{\prod_{b=1,b\neq a}^{n}(\sigma_a-\sigma_b)} \;\to\; \frac{\lambda_\star^\alpha(\sigma_a)\widetilde{\lambda}^{\dot{\alpha}}(\sigma_a)}{\prod_{b=1,b\neq a}^{n-1}(\sigma_a-\sigma_b)} \qquad a\in\{1,2\ldots n-1\}\,, \tag{C.8}$$

which is the same set of equations as Eq. (C.4) with $n-1$ particles and $\det[\lambda_\star^\alpha(z)] = d-1$. The number of solutions for Eq. (C.8) is $\mathcal{N}_{n-1,d-1}$ according to our notation. Next, we come study that for each solution of Eq. (C.8), how many solutions are there for σ_n. In the soft limit, the spinor helicity form of k_n is:

$$(k_n)^{\alpha\dot{\alpha}} = \epsilon(\lambda_n)^\alpha(\widetilde{\lambda}_n)^{\dot{\alpha}}\,.$$

[2] This is the number of the degrees of freedom in the external data $\{k_a\}$.

Since we have assumed in Eq. (C.6) that $(k_n)^{\alpha\dot{\alpha}} \to 0$ is driven by $\lambda^\alpha(\sigma_n) \to 0$, we must have $\lambda^\alpha(\sigma_n) \propto \epsilon(\lambda_n)^\alpha$ and

$$(\widetilde{\lambda}_n)^{\dot{\alpha}} \propto \widetilde{\lambda}^{\dot{\alpha}}(\sigma_n) \quad \Longleftrightarrow \quad \langle \widetilde{\lambda}_n \widetilde{\lambda}(\sigma_n) \rangle = 0 \,. \tag{C.9}$$

Since $\det[\widetilde{\lambda}(z)] = n-d-2$, Eq. (C.9) will give $n-d-2$ solutions for σ_n. Therefore, the number of solutions that lead to $\lambda^\alpha(\sigma_n) \to 0$ at the soft limit is $(n-d-2)\mathcal{N}_{n-1,d-1}$. Similarly analysis applies to the case $\widetilde{\lambda}^{\dot{\alpha}}(\sigma_n) \to 0$. We can write $\widetilde{\lambda}^{\dot{\alpha}}(z) \to (z-\sigma_n)\widetilde{\lambda}^{\dot{\alpha}}_\star(z)$ at the soft limit, which leads to:

$$(k_a)^{\alpha\dot{\alpha}} \;\to\; \frac{\lambda^\alpha(\sigma_a)\widetilde{\lambda}^{\dot{\alpha}}_\star(\sigma_a)}{\prod_{b=1,b\neq a}^{n-1}(\sigma_a-\sigma_b)} \qquad a \in \{1,2\ldots n\} \,. \tag{C.10}$$

Since now $\deg[\lambda^\alpha(z)] = d$, the solutions to this equation are in the sector $\mathcal{N}_{n-1,d}$. Then for each solution of Eq. (C.10), the equation for σ_n is $[\lambda_n \lambda(\sigma_n)] = 0$, which gives d solutions since $\lambda(z)$ has degree d. Collecting both contributions, we have the following recursive relation for $\mathcal{N}_{n,d}$:

$$\mathcal{N}_{n,d} = (n-2-d)\mathcal{N}_{n-1,d-1} + d\mathcal{N}_{n-1,d} \,. \tag{C.11}$$

Now $\mathcal{N}_{n,d}$ can be completely determined if we can fix the initial condition at $n=4$. First, since we do not allow $\deg[\lambda(z)] = 0$ or $\deg[\widetilde{\lambda}(z)] = 0$, which makes all Mandelstam variables vanish, we have $\mathcal{N}_{4,0} = \mathcal{N}_{4,2} = 0$. For $d=1$, we can write down the following equations:

$$\begin{aligned}
(\lambda_1)_\alpha(\widetilde{\lambda}_1)_{\dot{\alpha}} &= \frac{(\rho_{\alpha 0}+\rho_{\alpha 1}\sigma_1)(\widetilde{\rho}_{\dot{\alpha}0}+\widetilde{\rho}_{\dot{\alpha}1}\sigma_1)}{\sigma_{12}\sigma_{13}\sigma_{14}} \\
(\lambda_2)_\alpha(\widetilde{\lambda}_2)_{\dot{\alpha}} &= \frac{(\rho_{\alpha 0}+\rho_{\alpha 1}\sigma_2)(\widetilde{\rho}_{\dot{\alpha}0}+\widetilde{\rho}_{\dot{\alpha}1}\sigma_2)}{\sigma_{21}\sigma_{23}\sigma_{24}} \\
(\lambda_3)_\alpha(\widetilde{\lambda}_3)_{\dot{\alpha}} &= \frac{(\rho_{\alpha 0}+\rho_{\alpha 1}\sigma_3)(\widetilde{\rho}_{\dot{\alpha}0}+\widetilde{\rho}_{\dot{\alpha}1}\sigma_3)}{\sigma_{31}\sigma_{32}\sigma_{34}} \,.
\end{aligned} \tag{C.12}$$

Then we take the spinor inner products and get:

$$s_{12} = \frac{\langle \widetilde{\rho}_0 \widetilde{\rho}_1 \rangle [\rho_0 \rho_1]}{\sigma_{13}\sigma_{14}\sigma_{23}\sigma_{24}} \qquad s_{23} = \frac{\langle \widetilde{\rho}_0 \widetilde{\rho}_1 \rangle [\rho_0 \rho_1]}{\sigma_{21}\sigma_{24}\sigma_{31}\sigma_{34}} \,, \tag{C.13}$$

under the $SL(2,\mathbb{C})$ gauge $\sigma_2 = 0$, $\sigma_3 = 1$ and $\sigma_4 = \infty$, we have:

$$-\frac{s_{12}}{s_{23}} = \frac{\sigma_{24}\sigma_{34}}{\sigma_{14}\sigma_{24}}\frac{\sigma_{12}}{\sigma_{32}} \;\to\; \sigma_1 \,. \tag{C.14}$$

Namely, we only have one solution $\sigma_1 = -s_{12}/s_{23}$, such that $\mathcal{N}_{4,1} = 1$. With this initial condition, the recursive relation generates an Eulerian number pattern:

$$\mathcal{N}_{n,d} = A(n-3, d-1)\,. \tag{C.15}$$

If we sum over the d sectors, we recover the correct counting for the total number of solutions to SE, because of the identity:

$$\sum_{d=1}^{n-3} A(n-3, d-1) = (n-3)!\,. \tag{C.16}$$

The first few Eulerian numbers series are shown in Table C.1 on the current page. This derivation was first given by Cachazo, He and Yuan in [32].

Table C.1. Eulerian numbers with $1 \leqslant n \leqslant 6$.

$A(n,m)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$\sum_{m=0}^{n-1} A(n,m)$
$n=1$	1	–	–	–	–	–	1
$n=2$	1	1	–	–	–	–	2
$n=3$	1	4	1	–	–	–	6
$n=4$	1	11	11	1	–	–	24
$n=5$	1	26	66	26	1	–	120
$n=6$	1	57	302	302	57	1	720

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